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ANALYSIS OF FLUID QUEUES USING LEVEL CROSSING METHODS

by

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ABSTRACT

This dissertation is concerned with the application of the level crossing method on fluid queues driven by a background process. The basic assumption of the fluid queue in this thesis is that during the busy period of the driving process, the fluid content fills at *net* rate r_1, and during the idle period of the driving process, the fluid content, if positive-valued, empties at a rate r_2. Moreover, nonempty fluid content, leaks continuously at a rate r_2. The fluid models considered are: the fluid queue driven by an $M/G/1$ queue in Chapter 2, the fluid queue driven by an $M/G/1$ queue with *net* input and leaking rate depending on fluid level, and *type of arrivals* in the driving $M/G/1$ queue, in chapter 3, and the fluid queue driven by an $M/G/1$ queue with upward fluid jumps in Chapter 4. In addition, a triangle diagram has been introduced in this thesis as a technique to visualize the proportion of time that the content of the fluid queue is increasing or decreasing during nonempty cycles. Finally, we provide several examples on how the probability density function of the fluid level is related to the probability density function of the waiting time of $M/G/1$ queues with different disciplines.

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LIST OF ABBREVIATIONS AND SYMBOLS

Following is a collection of the abbreviations and notations used frequently and consistently throughout the dissertation.

Abbreviation	**Description**
cdf	Cumulative distribution function
ccdf	Complementary cumulative distribution function
eq	Equation
LC	Level-crossing method
pdf	Probability distribution function

Symbol	**Description**
$\mathcal{A}$	Activity period of fluid queue
$\mathcal{A}_T$	Duration of activity periods of fluid queue in a wet cycle
B	Busy period of driving queue
$B(x)$	Cumulative distribution function of $M/G/1$ queue busy period
B_c	Busy cycle of driving queue
$\mathcal{C}$	Fluid level of fluid queue
$D_t(x)$	Number of downcrossings of level x during $(0, t)$
$Exp(\lambda)$	Exponential random variable with parameter λ
$\mathcal{E}$	Empty period of fluid queue
I	Idle period of driving queue
λ	Arrival rate of driving queue

$\mathcal{N}_p$	Number of peaks within a fluid wet cycle
$\phi(t)$	States of the fluid queue
$Poiss(\lambda)$	Poisson random variable with parameter λ
r_1	Net input rate for fluid queue
r_2, r_3	Leaking rates for fluid queue
$\mathcal{S}$	Silence period of fluid queue
$\mathcal{S}_T$	Duration of silence periods of fluid queue in a wet cycle
$U_t(x)$	Number of upcrossings of level x during $(0, t)$
μ	Service rate for driving queue
W	Waiting time of an $M/G/1$ queue
$\mathcal{W}$	Wet period of fluid queue
$\mathcal{W}_c$	Wet cycle of fluid queue
$\mathcal{W}(x)$	Cumulative distribution of the fluid queue wet period

Chapter 1

Introduction

1.1 Introduction

Waiting in line is a phenomenon in daily life. All of us encounter certain types of waiting lines before receiving service, such as waiting lines at Tim Hortons to buy a coffee, in a hospital (or outside a hospital) to access a healthcare service, at the border to enter another country, or on the phone when calling customer support. Waiting in line is annoying for most of us; unfortunately, no one can escape waiting in line. However, we can use a mathematical model to study the characteristics of waiting lines to reduce the delay time and to improve the efficiency of the service.

A mathematical model used to study the characteristics of waiting time phenomena is called a queueing model and was first studied by Erlang [17] in 1909. The original paper by Erlang was intended to study traffic in telecommunications. Since then, hundreds of queueing papers are published each year. In traditional queueing theory, a queueing model consists of a single server that provides service to individual arrivals. The inter-arrival times are arbitrary, and the service times required are uncertain. Arrivals may or may not need to wait in line before receiving service. For this single server queueing model, we are interested in the experience of arrivals in terms of delay and service time. The assumptions of queueing models can be summarized in the notation $A/B/n/m$ introduced by Kendall [23], where A and B correspond to the distribution of interarrival time and service time, n is the number of servers for the queueing model, and m is the capacity of the system.

The queueing models we have mentioned so far can be classified as *discrete state space* queueing models where the states are the numbers of customers in the system. In discrete state space queueing models, some characteristics of the queueing model can be analyzed by a discrete event. For instance, using the fact that the waiting time and service time are attached on individual arrivals, we can find the distribution of the busy period by adding up each individual's service time in a busy cycle, where the busy cycle is defined as the time between of two consecutive arrivals' waiting times being equal to 0.

Recently, *continuous state space* queueing models, such as the fluid queue, have received more attention from researchers. In a fluid queue, arrivals are too small to distinguish between them in the fluid system. Thus, it is easier to consider arrivals as continuous fluid that enters and exits the fluid queue. One example of a fluid queue has arrivals which enter and leave the queue with rates modulated by a background Markovian random environment process such as a birth-death processes, $G/G/1$ queue, or vacation queueing model ([42]). Researchers usually refer to capacity of the queue as the fluid content. In this dissertation, we focus on infinite-capacity and finite fluid content queues driven by an $M/G/1$ queue or modified $M/G/1$ queues.

1.2 Literature review of fluid queues driven by Markovian process

1.2.1 Stationary distribution of fluid queue

Consider an infinite fluid content queue with a background Markovian random process (or environment process) with finite state space. The fluid content fills and empties at rates that are modulated by the background Markovian random process, which is not necessary observable. Denote by $\mathcal{C}(t)$ the fluid level at time t (≥ 0), and by $Z(t) \in \{1, 2, 3, \ldots\}$ the states of the continuous time random process which makes the joint process $\{\mathcal{C}(t), Z(t)\}_{t\geq 0}$ a Markov process. The characteristics of the fluid queue can be summarized as follows:

$$\frac{d\mathcal{C}(t)}{dt} = \begin{cases} r_{Z(t)}, & \mathcal{C}(t) > 0, \\ \max(0, r_{Z(t)}), & \mathcal{C}(t) = 0. \end{cases} \tag{1.1}$$

where $r_{Z(t)} < \infty$ is the *net* input rate of the fluid queue at time t. Using Kulkarni's terminology [25], we may refer to the $\{\mathcal{C}(t)\}_{t\geq 0}$ process as a fluid process driven by the background Markov renewal process $\{Z(t)\}_{t\geq 0}$.

Consider a Markov renewal process $\{Z(t)\}_{t\geq 0}$ with the states of an $M/G/1$ queue defined as follows

$$Z(t) = \begin{cases} 1 & \text{if the server of the } M/G/1 \text{ queue is busy,} \\ 2 & \text{if the server of the } M/G/1 \text{ queue is idle.} \end{cases} \tag{1.2}$$

The $\{\mathcal{C}(t)\}_{t\geq 0}$ process is referred to as a fluid queue driven by an $M/G/1$ queue [4, 44]. Virtamo and Norros [44] study the steady-state distribution of the fluid level using a spectral decomposition technique, where the service time of the driving queue is exponentially distributed with rate μ (i.e. the net input rate of the fluid queue is modulated by an $M/G/1$ queue). Adan and Resing [4] re-investigate the model and derive the Laplace-Stieltjes transform (henceforth LST) of the steady-state distribution of the level of fluid queue with net input rate $r_1 = 1$ and leaking rate $r_2 = -1$ driven by an $M/G/1$ queue, using a regenerative processes approach[1].

More precisely, let $\{B_n\}_{n\geq 1}$ be a sequence of busy periods, and $\{I_n\}_{n\geq 1}$ be a sequence of idle periods of an $M/G/1$ queue (see ([45]) for definition of busy and idle periods). Adan and Resing consider $B_1, I_1, B_2, I_2, \ldots$ as an alternating renewal process, the fluid level $\mathcal{C}_n$ at the beginning of the nth busy period B_n satisfies the well known Lindley recursion (see [12, p. 3 eq (1.1)] and [18, p.14, eq (1.5)] for the details on Lindley recursion)

$$\mathcal{C}_{n+1} = (0,\ \mathcal{C}_n + B_n - I_n)^+, \quad n = 1, 2, \ldots. \tag{1.3}$$

Observation 1.2.1. *Consider $\mathcal{C}_n$ to be the waiting time of the n^{th} arrival, B_n to be the service time of the n^{th} arrival, and I_n to be the interarrival time between n^{th} and $(n+1)^{st}$ arrivals of an $M/G/1$ queue, then (1.3) is the Lindley recursion for an virtual waiting time of the $M/G/1$ queue. Thus, expression (1.3) suggests that the characteristics of the fluid queue can be achieved by understanding the characteristics of the $M/G/1$ queue.*

Laplace-Stieltjes transform of steady-state distribution of the fluid level

Let $\widetilde{B}(s) := \int_0^\infty e^{-sx} d\mathbb{P}[B \leq x]$ be the LST of the probability density function (pdf) of the busy period of an $M/G/1$ queue which is driving the fluid queue, and $\widetilde{\mathcal{C}}(s) := \int_{0^-}^\infty e^{-sx} d\mathbb{P}[\mathcal{C} \leq x]$ be the LST[2] of the pdf of the fluid level of the fluid queue $\mathcal{C}$. Adan and Resing derive the LST of the pdf of the fluid level of the fluid queue driven by an

[1] It turns out that using renewal theory, the expected fluid cycle and the expected number of peaks within a fluid cycle can be easily achieved.

[2] We use 0^- in the lower limit of the integral of the LST to indicate that there has a point mass at level 0 for the fluid level $\mathcal{C}$.

$M/G/1$ queue in [4, eq. (3), p. 172] as

$$\widetilde{\mathcal{C}}(s) = \left(\frac{1-\lambda\mathbb{E}[B]}{1+\lambda\mathbb{E}[B]}\right)\left(\frac{s+\lambda-\lambda\widetilde{B}(s)}{s-\lambda+\lambda\widetilde{B}(s)}\right), \quad s > 0, \tag{1.4}$$

and the tail probability of the fluid level of the fluid queue driven by an $M/M/1$ queue [4, eq. (5), p. 173],

$$\mathbb{P}[\mathcal{C} > x] = 4\rho e^{-(\mu/2-\lambda)x} - 2\rho\left[\int_0^x e^{-(\mu/2-\lambda)(x-y)} f(y)dy + 1 - \int_0^x f(y)dy\right] \tag{1.5}$$

for $x \geq 0$, where $1/\mu$ is the expected service time in the $M/M/1$ system, with a $\rho := \lambda/\mu$ and a point mass at level 0,

$$\pi_{\mathcal{E}} = 1 - \lim_{x\to 0^+} \mathbb{P}[\mathcal{C} > x] = 1 - 2\lambda/\mu. \tag{1.6}$$

Formula (1.6) coincides with the results derived by Virtamo and Norros in [44] using a spectral decomposition approach.

Applying Leibniz's Rule [14, Theorem 2.4.1, p. 69] to (1.5), the pdf of the fluid level can be found as

$$f(x) = 4\left(\frac{\lambda}{\mu}\right)(\mu/2-\lambda)\, e^{-(\mu/2-\lambda)x} - 2\left(\frac{\lambda}{\mu}\right)\left(\frac{\mu}{2}-\lambda\right)\int_0^x e^{-\left(\frac{\mu}{2}-\lambda\right)(x-y)} f(y)dy \tag{1.7}$$

for $x > 0$.

Observation 1.2.2. *Letting $x \to 0^+$ in (1.7) yields*

$$\lim_{x\to 0^+} f(x) = 2\lambda\left(1 - 2\lambda/\mu\right). \tag{1.8}$$

As shown in (1.6), the term $(1-2\lambda/\mu)$ on the right-hand side of (1.8) is the point mass of fluid at level 0; the term 2λ will be explained later in Chapter 2.

1.2.2 New contributions

Some researchers (e.g. [38]) have considered (matrix analytic) Erlangized fluid queues. Other researchers have addressed fluid queues driven by an $M/G/1$ queue (e.g. [4, 25, 44]). In Chapter 2 of this dissertation, we add a level crossing approach and reanalyze fluid queue models. The additional value of doing this is that the new approach simplifies the derivation and adds to the understanding of the geometry of results in this field. Further, the level crossing method allows for generalizations to models that would be difficult to analyze with other methods. In Chapter 3, a new model is introduced for which the leaking rate is level-dependent. Also, another model considers a situation in which the leaking rate depends on the type of arrival. In Chapter 4, a new model is introduced where arrivals are accepted with a fixed probability, but regardless of whether the arrivals are accepted or not, the fluid content increases. Another contribution is the introduction of a triangle diagram to help illustrate properties of fluid queues.

1.2.3 Potential applications

Fluid queues arise in financial applications, health and pharmacokinetic applications, battery recharge, water resources, and insurance. In finance, the value of a portfolio moves continuously according to information over time. In health, the glucose reading of a diabetic increases or decreases continuously with food intake and exercise. Many devices use rechargeable batteries. The power level decreases over time but the battery can be recharged as needed. In insurance, the surplus has an upper bound called a barrier and the company may pay dividends when the barrier is reached. All of these situations can be modeled by fluid queues.

1.3 Elementary properties of renewal processes

Before starting to introduce the phenomenon of a rate balance equation, we first introduce some properties of renewal processes.

Suppose that $(\Omega, \mathbb{F}, \mathbb{P})$ is a probability space. A real-valued function X is said to be a random variable defined on $(\Omega, \mathbb{F}, \mathbb{P})$ if

$$X : \Omega \to \mathbb{R}. \tag{1.9}$$

A stochastic process is a dynamic version of random variable. For more details, refer to Ross [34, Section 2.8, pp. 83 - 85], Taylor and Karlin [40, Section 1.1, p. 5], and Wolff [45, Section 2.1, pp. 53 - 54]. Suppose that $T = [0, \infty)$, such that

$$\forall t \in T, \quad X_t : \Omega \to \mathbb{R} \tag{1.10}$$

is a random variable defined on $(\Omega, \mathbb{F}, \mathbb{P})$. For $t \in T$ and $\omega \in \Omega$, the mapping

$$X(t, \omega) : T \times \Omega \to \mathbb{R} \tag{1.11}$$

which is jointly measurable in (t, ω), is called a stochastic process with indexing set T. The index t is often interpreted as time, and we refer to $X(t, \omega)$ as the state of the stochastic process at time $t \in T$.

Remark 1.3.1. *Even though we obtain (1.10) by replacing X by X_t in equation (1.9), we assume that t is a fixed number. By knowing $\omega \in \Omega$, we know the exact value of $X_t(\omega)$ in (1.10); therefore, $X_t(\omega)$ is a mapping of $\omega \in \Omega$, and it is a random variable. The mapping (1.11) is a function of $t \in T$ and $\omega \in \Omega$. If we fix $\omega_1 \in \Omega$, then $X(t, \omega_1)$ is a function of t. Thus for each $\omega \in \Omega$, we have a set of random variables X_t, which we* *call a stochastic process. For simplification of notation, we will use $X(t) := X(t,\omega)$ for a particular $\omega \in \Omega$ and $t \in T$ [30].*

Definition 1.3.2 (Ross, 2003, Counting processes, p. 288). *A stochastic process $\{N(t)\}_{t\geq 0}$ is a counting process if $N(t)$ represents the number of events that occurred in the interval $(0, t]$.*

Definition 1.3.3 (Ross, 2003, Renewal processes, p. 401). *A stochastic process $\{N(t)\}_{t\geq 0}$ is a renewal process if $\{N(t)\}_{t\geq 0}$ is a counting process and the interarrival time for the events are independent and identically distributed (henceforth iid) with expected value μ.*

Proposition 1.3.4 (Ross, 2003, Proposition 7.1 , p. 407). *If $\{N(t)\}_{t\geq 0}$ is a renewal process, then with probability 1 (w.p.1),*

$$\lim_{t\to\infty} \frac{N(t)}{t} \to \frac{1}{\mu}.$$

Theorem 1.3.5 (Ross, 2003, Elementary Renewal Theorem, p. 409). *If $\{N(t)\}_{t\geq 0}$ is a renewal process, then*

$$\lim_{t\to\infty} \frac{E[N(t)]}{t} \to \frac{1}{\mu}.$$

Theorem 1.3.6 (Ross, 2003, Proposition 7.3, Renewal reward theorem, pp. 416 - 417). *Let $\{N(t)\}_{t\geq 0}$ be a renewal process having interarrival times $X_n, n \geq 1$, with $\mathbb{E}[X_n] = \mathbb{E}[X] < \infty$, and R_n, $n \geq 1$, be the reward earned at the time of the nth renewal. Assume that R_n are independent and identically distributed with $\mathbb{E}[R_n] = \mathbb{E}[R] < \infty$. Define $R(t)$ be the total reward earned by time t, i.e.*

$$R(t) = \sum_{n=1}^{N(t)} R_n. \tag{1.12}$$

Then, we have

$$\lim_{t\to\infty} \frac{R(t)}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}, \quad \textit{(with probability 1)}, \tag{1.13}$$

$$\lim_{t\to\infty} \frac{\mathbb{E}[R(t)]}{t} = \frac{\mathbb{E}[R]}{\mathbb{E}[X]}. \tag{1.14}$$

Let $\{X(t,\omega)\}_{\{t\in T,\ \omega\in\Omega\}}$ be a continuous-time Markov chain with a *countable* discrete state space of S such tat the states of the future of the process given the present state is independent of the past. We define a sample path of $X(t,\omega)$ as a mapping of $t \in T$, such that

$$t \to X(t,\omega) := X(t) \tag{1.15}$$

for a particular choice of $\omega \in \Omega$. In addition, we assume that (1.15) is a right-continuous step function. The level-crossing method (henceforth LC) method focuses on the behaviour of the sample path of a stochastic process, especially the number of times that the leading point (henceforth LP) of the sample path enters and leaves certain states. By studying the SP, the LC method allows researchers to write an integral equation for the steady-state pdf of $X(t)$ as $t \to \infty$.

Definition 1.3.7 (Brill, 2008, Definition 2.1, p. 19)**.** *A function*

$$X : T \to S$$

is a sample path of a stochastic process if $X(t)$ is a bounded real-valued cadlag function[3] *[8, p. 121] for all $t > 0$.*

[3] A function is a cadlag function, if it is right continuous with left limits.

For a fixed x, let t_k be a time point in $(0, t)$, and define

$$D_t(x) := \#\{k : X(t_k^-) \geq x \text{ and } X(t_k) < x\}, \tag{1.16}$$

$$U_t(x) := \#\{k : X(t_k^-) \leq x \text{ and } X(t_k) > x\}, \tag{1.17}$$

where $X(t)$ is a sample path of the stochastic process [12, pp. 31-32]. We interpret $D_t(x)$ and $U_t(x)$ as the number of downcrossings and upcrossings, respectively, of level x in $(0, t)$. Further, since $D_t(x)$ and $U_t(x)$ represent the number of events (downcrossings and upcrossings of level x) in $(0, t)$, $\{U_t(x)\}_{t\geq 0}$ and $\{D_t(x)\}_{t\geq 0}$ are counting processes.

Remark 1.3.8. *Formulas (1.16) and (1.17) require that the left limits of the sample path, $X(t)$, exist for all $t > 0$.*

Theorem 1.3.9 (Brill, 2008, Theorem 3.3, p. 54)**.** *Given that $D_t(x)$ is a renewal process of downcrossings of level x of an $M/G/1$ queue, we have*

$$\lim_{t\to\infty} \frac{E[D_t(x)]}{t} \overset{a.s.}{=} \lim_{t\to\infty} \frac{D_t(x)}{t} \overset{a.s.}{=} f(x)$$

where $f(x)$ is the steady-state probability density function of the state variable.

Theorem 1.3.10 (Brill, 2008, principle of rate balance, pp. 34 - 35)**.** *If $\{D_t(x)\}_{t\geq 0}$ and $\{U_t(x)\}_{t\geq 0}$ are counting processes for the number of downcrossings and upcrossings of level x respectively in an $M/G/1$ queue, then*

$$\lim_{t\to\infty} \frac{D_t(x)}{t} \overset{a.s.}{=} \lim_{t\to\infty} \frac{U_t(x)}{t},$$

$$\lim_{t\to\infty} \frac{E[D_t(x)]}{t} = \lim_{t\to\infty} \frac{E[U_t(x)]}{t}.$$

For the proof of Theorem 1.3.9 and 1.3.10, refer to Brill [12, p. 54 for Theorem 1.3.9 and pp. 34 - 35 for Theorem 1.3.10].

1.4 Applications of level crossing methods on fluid queues

1.4.1 Example 1: Buffer level of fluid queue driven by a single On/Off source

Let $\{\mathcal{C}(t)\}_{t\geq 0}$ be the fluid level at time t, and $M(t) \in \{0,1\}$ be the state of the background process that alternates between on and off periods. During the on periods, $M(t) = 1$, the fluid content fills at *net* rate $c_1(x) > 0$. During the off periods, $M(t) = 0$, the fluid content empties at rate $c_0(x) > 0$ if $\mathcal{C}(t) > 0$, or $c_0(x) = 0$ if $\mathcal{C}(t) = 0$, where x is the level of fluid content at time t. It is assumed that the on and off periods are exponentially distributed with rate μ and λ respectively.

Define $\{\mathcal{C}(t), M(t)\}_{t\geq 0}$ as a two-dimensional Markov process. Following the argument in [12, Section 10.10.1], the partial steady-state cumulative distribution function (cdf) of fluid level is

$$\mathbb{F}_i(x) = \lim_{t\to\infty} \mathbb{P}[\mathcal{C}(t) \leq x, M(t) = i], \quad x > 0,\ i = 0,1, \tag{1.18}$$

and the partial pdf of the fluid level is

$$f_i(x) = \frac{d}{dx}\mathbb{F}_i(x), \quad x > 0,\ i = 0,1, \tag{1.19}$$

whenever the derivatives exist. The marginal cdf of the fluid level is

$$\mathbb{P}[\mathcal{C}(t) \leq x] = \pi_{\mathcal{E}} + \mathbb{F}_0(x) + \mathbb{F}_1(x), \quad x \geq 0, \tag{1.20}$$

where

$$\pi_{\mathcal{E}} = \lim_{t\to\infty} \mathbb{P}[\mathcal{C}(t) = 0] = 1 - \int_{0^+}^{\infty} f(x)dx. \tag{1.21}$$

and

$$f(x) = f_0(x) + f_1(x), \quad x > 0. \tag{1.22}$$

For a small interval of length Δt, we have

$$\begin{aligned}\mathbb{F}_0(t+\Delta t, x) = &\,\mathbb{F}_0(t, x + c_0(x^*)\Delta t)\left(1 - \lambda\Delta t + o(\Delta t)\right)\\ &+ \mathbb{F}_1(t, x - c_1(x^*)\Delta t)\left(\mu\Delta + o(\Delta t)\right),\end{aligned} \tag{1.23}$$

$$\begin{aligned}\mathbb{F}_1(t+\Delta t, x) = &\,\mathbb{F}_0(t, x + c_0(x^*)\Delta t)\left(\lambda\Delta t + o(\Delta t)\right)\\ &+ \mathbb{F}_1(t, x - c_1(x^*)\Delta t)\left(1 - \mu\Delta + o(\Delta t)\right),\end{aligned} \tag{1.24}$$

where $x < x^* < x + c_0(x)\Delta t$. Formula (1.23) follows since in order for the system to stay in $[0, x]$ at time $t + \Delta t$, it is necessary and sufficient that the system is in $[0, x + c_0(x^*)\Delta t]$ at time t and the off period with rate λ does not end during $(t, t + \Delta t)$, or the system is in $[0, x - c_0(x^*)\Delta t]$ at time t and the on period with rate μ ended during $(t, t + \Delta t)$. For the special case when the state happens to be at level 0 at time $t + \Delta t$, by the memoryless property of the exponential distribution, the system state is at level 0 for a time $\sim Exp(\lambda)$. Formula (1.24) can be interpreted in a similar manner as (1.23).

Adding and subtracting $F_0(t, x)$ to (1.23) and $F_1(t, x)$ to (1.24), dividing both equations by Δt, and letting $\Delta t \to 0$, we get *Chapman-Kolmogorov equations* that are satisfied by the two-dimensional Markov process (see [33, p. 74] and [19, p. 230] for more details):

$$\frac{1}{c_0(x)}\frac{\partial}{\partial t}F_0(t,x) - \frac{\partial}{\partial x}F_0(t,x) = -\lambda\frac{F_0(t,x)}{c_0(x)} + \mu\frac{F_1(t,x)}{c_0(x)}, \tag{1.25}$$

$$\frac{1}{c_1(x)}\frac{\partial}{\partial t}F_1(t,x) + \frac{\partial}{\partial x}F_1(t,x) = \lambda\frac{F_0(t,x)}{c_1(x)} - \mu\frac{F_1(t,x)}{c_1(x)}. \tag{1.26}$$

Taking limits as $t \to \infty$, we get

$$c_0(x)f_0(x) + \mu\int_0^x f_1(y)dy = \lambda\int_0^x f_0(y)dy, \tag{1.27}$$

$$c_1(x)f_1(x) + \mu\int_0^x f_1(y)dy = \lambda\int_0^x f_0(y)dy, \tag{1.28}$$

which yields the following equality

$$c_0(x)f_0(x) = c_1(x)f_1(x). \tag{1.29}$$

Equation (1.29) can be interpreted as total downcrossing rate equals total upcrossing rate of level x.

Remark 1.4.1 (Brill, 2008, pp. 433 - 437, *page method*)**.** *Equations (1.27) and (1.28) have interesting interpretations in terms of rate into and rate out of the composite states* $\{(0,x],0\}$ *and* $\{(0,x],1\}$.

For equation (1.27): Define LP to be the leading point of a SP (sample path). Then

$$c_0(x)f_0(x) + \mu\int_0^x f_1(y)dy, \quad x > 0 \tag{1.30}$$

is the downcrossing rate into $\{(0,x],0)\}$. *For instance, the first term in (1.30) corresponds to the downcrossing rate of level* x, *and the second term corresponds to the rate* *at which on periods end when the state is in* $\{(0,x],1\}$*, resulting in parallel transitions into* $\{(0,x],0\}$*. On the other hand, the rate at which off periods end when the state is in* $\{(0,x],0\}$ *is*

$$\lambda \int_0^x f_0(y)dy, \quad x > 0, \tag{1.31}$$

resulting in parallel transitions out of $\{(0,x],0\}$*, and it is the only way to exit* $\{(0,x],0\}$*. Thus, the expression (1.31) is the total rate out of* $\{(0,x],0\}$*. Balancing the total rate out of and rate into* $\{(0,x],0\}$ *yields equation (1.27). Similar arguments yield equation (1.28).*

Remark 1.4.1 suggests that the LC method can be used to bypass using the *Chapman-Kolmogorov equations*. The solution of (1.27) and (1.28) can be found in [12, pp. 436-437, eq. (10.82) and eq. (10.84)], namely

$$f(x) = \lambda \left(\frac{c_0(x) + c_1(x)}{c_0(x)c_1(x)} \right) \pi_{\mathcal{E}} \exp\left(- \int_0^x \left(\frac{\mu}{c_1(y)} - \frac{\lambda}{c_0(y)} \right) dy \right), \tag{1.32}$$

where $\pi_{\mathcal{E}}$ can be achieved by using the normalizing condition. (1.21), We have

$$\pi_{\mathcal{E}} = \left(1 + \lambda \int_0^\infty \left(\left(\frac{c_0(x) + c_1(x)}{c_0(x)c_1(x)} \right) \exp\left(- \int_0^x \left(\frac{\mu}{c_1(y)} - \frac{\lambda}{c_0(y)} \right) dy \right) \right) dx \right)^{-1}. \tag{1.33}$$

Note that the solutions (1.32) and (1.33) in [12, pp. 443 - 440] are for 'a dam with alternating influx and efflux.' It turns out that the solutions are for the underlying fluid queue as well. Let $c_0(x) = c_1(x) = 1$. Substituting $c_0(x)$ and $c_1(x)$ into (1.32) and (1.33) yields

$$f(x) = 2\lambda \pi_{\mathcal{E}} e^{-(\mu-\lambda)x}, \quad x > 0, \tag{1.34}$$

and

$$\pi_{\mathcal{E}} = 1 - \frac{2\lambda}{\mu + \lambda}. \tag{1.35}$$

Note that by letting $c_0(x) = c_1(x) = 1$, the expression (1.34) is similar to the waiting time distribution of a modified $M/M/1$ queue where first arrivals in a cycle have arrival rate 2λ and other arrivals have arrival rate λ.

To see this, denote λ and μ as the arrival rate of general arrivals, μ be the service rate of all arrivals, and $f(x)$ be the pdf of waiting time of the modified $M/M/1$ queue. Using an LC argument, we have for $x > 0$

$$f(x) = 2\lambda\pi_0 e^{-\mu x} + \lambda \int_0^x e^{-\mu(x-y)} f(y)dy, \tag{1.36}$$

where π_0 is the probability of the $M/M/1$ queue being empty. Differentiating both sides of (1.36) with respect to x yields

$$\begin{aligned} f'(x) &= -2\lambda\mu\pi_0 e^{-\mu x} - \lambda \int_0^x \mu e^{-\mu(x-y)} f(y)dy + \lambda f(x) \\ &= -(\mu - \lambda) f(x), \quad x > 0 \end{aligned} \tag{1.37}$$

with solution

$$f(x) = Ae^{(\mu-\lambda)x}, \quad x > 0. \tag{1.38}$$

Let $x \to 0$ in (1.36) and (1.38) and compare the coefficients to yield

$$A = 2\lambda\pi_0. \tag{1.39}$$

Letting $\pi_0 = \pi_{\mathcal{E}}$ and substituting A in (1.39) into (1.38) yields

$$f(x) = 2\lambda\pi_{\mathcal{E}}e^{-(\mu-\lambda)x}, \quad x > 0. \tag{1.40}$$

which is identical to $f(x)$ defined in (1.34). Expressions (1.34) and (1.40) suggests that we can study the characteristics of the fluid queue by studying the characteristics of the driving queueing model or vice versa.

1.4.2 Example 2: Distribution of the fluid level of the fluid queue with a finite capacity $\mathcal{R}$

Similar to the $M/G/1$ queue, not all queueing models have a potentially infinite waiting time, such as 'queue with uniformly bounded actual waiting time' (henceforth finite $M/G/1$ with upper boundary), [15, p. 478]. Researchers are interested in the finite content fluid queue as well. Denote by $\mathcal{R}$ the upper limit of the fluid content, and by $\mathcal{C}^{(\mathcal{R})}(t)$ the fluid level at time t. The characteristic of the finite content fluid queue can be summarized as follows

$$\frac{d\mathcal{C}^{(\mathcal{R})}(t)}{dt} = \begin{cases} r_{Z(t)}, & Z(t) = 1,\ 2;\ 0 < \mathcal{C}^{(\mathcal{R})}(t) < \mathcal{R}, \\ \max(0, r_{Z(t)}), & Z(t) = 2;\ \mathcal{C}^{(\mathcal{R})}(t) = 0, \end{cases} \tag{1.41}$$

where $Z(t)$ is defined in (1.2). However, when the fluid level reaches the maximum capacity of the fluid content (i.e. $\mathcal{C}^{(\mathcal{R})}(t) = \mathcal{R}$), all customers in the driving $M/G/1$ queue are lost, and the server of the driving $M/G/1$ queue becomes idle. The driving queue can be thought of as an $M/G/1$ queue with disasters (or negative arrivals), once the disaster arrive to the queue (here, in our case, the fluid level of the fluid queue hits level $\mathcal{R}$), all customers in the system are lost (for the details of the $M/G/1$ queue with negative arrivals or disasters, see [27]).

Consider a fluid queue with a finite capacity $\mathcal{R}$ driven by an $M/G/1$ queue defined in (1.41). Let $\mathcal{C}^{(\mathcal{R})}$ be the steady-state random variable of $\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\geq 0}$ as t goes to ∞. During the busy period of an $M/G/1$ queue, the content fills at *net* rate 1, and during the idle period, the content empties at rate 1 as long as the fluid level is above zero. The fluid queue with finite capacity $\mathcal{R} > 0$ is characterized as follows: the fluid queue with upper boundary $\mathcal{R}$ is equivalent to the fluid queue *without upper boundary* driven by an $M/G/1$ queue except that once the fluid level reaches level $\mathcal{R}$, all customers in the driving $M/G/1$ queue are lost. Here, we refer to the fluid queue without upper boundary discussed in Section 1.2.1 as an ordinary fluid queue.

Remark 1.4.2. *There are two possible ways for the driving $M/G/1$ queue becomes empty. First, all customers are served during the busy period of the driving $M/G/1$ queue, before the fluid level hits $\mathcal{R}$. This is a normal completion for busy periods. Second, the fluid level reaches level $\mathcal{R}$ such that all customers in the driving $M/G/1$ are required to leave the $M/G/1$ queue instantly.*

Denote by $\{\mathcal{C}(t)\}_{t\geq 0}$ and $\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\geq 0}$ the fluid level for the ordinary fluid queue and finite fluid queue respectively. To illustrate the differences between the infinite and finite fluid queues, we plot the states of the driving $M/G/1$ queue $\{Z(t)\}_{t\geq 0}$ corresponding to the ordinary fluid queue, and the sample paths of the fluid level of the infinite and finite fluid queues in Figure 1.1 (a), (b), and (c) respectively. As observed in the Figures 1.1 (b) and (c), on one hand, once the fluid level $\{\mathcal{C}(t)\}_{t\geq 0}$ of the finite fluid queue reaches level $\mathcal{R}$, the fluid level starts (instantly) to decrease at rate -1. On the other hand, the fluid level $\mathcal{C}(t)\}_{t\geq 0}$ of the ordinary fluid queue can go beyond level $\mathcal{R}$. Denote by B the busy period of the driving $M/G/1$ queue of the finite fluid queue, it is important to note that $\mathbb{P}[B \leq \mathcal{R}] = 1$ as the consequence of the upper boundary of the fluid level $\mathcal{R}$.

As observed in Figure 1.1, the duration of the time that the fluid level $\{\mathcal{C}(t)\}_{t\geq 0}$ of the ordinary fluid queue below level $\mathcal{R}$ is $G_1 + G_2 + G_3 + G_4$ which equals to the duration of

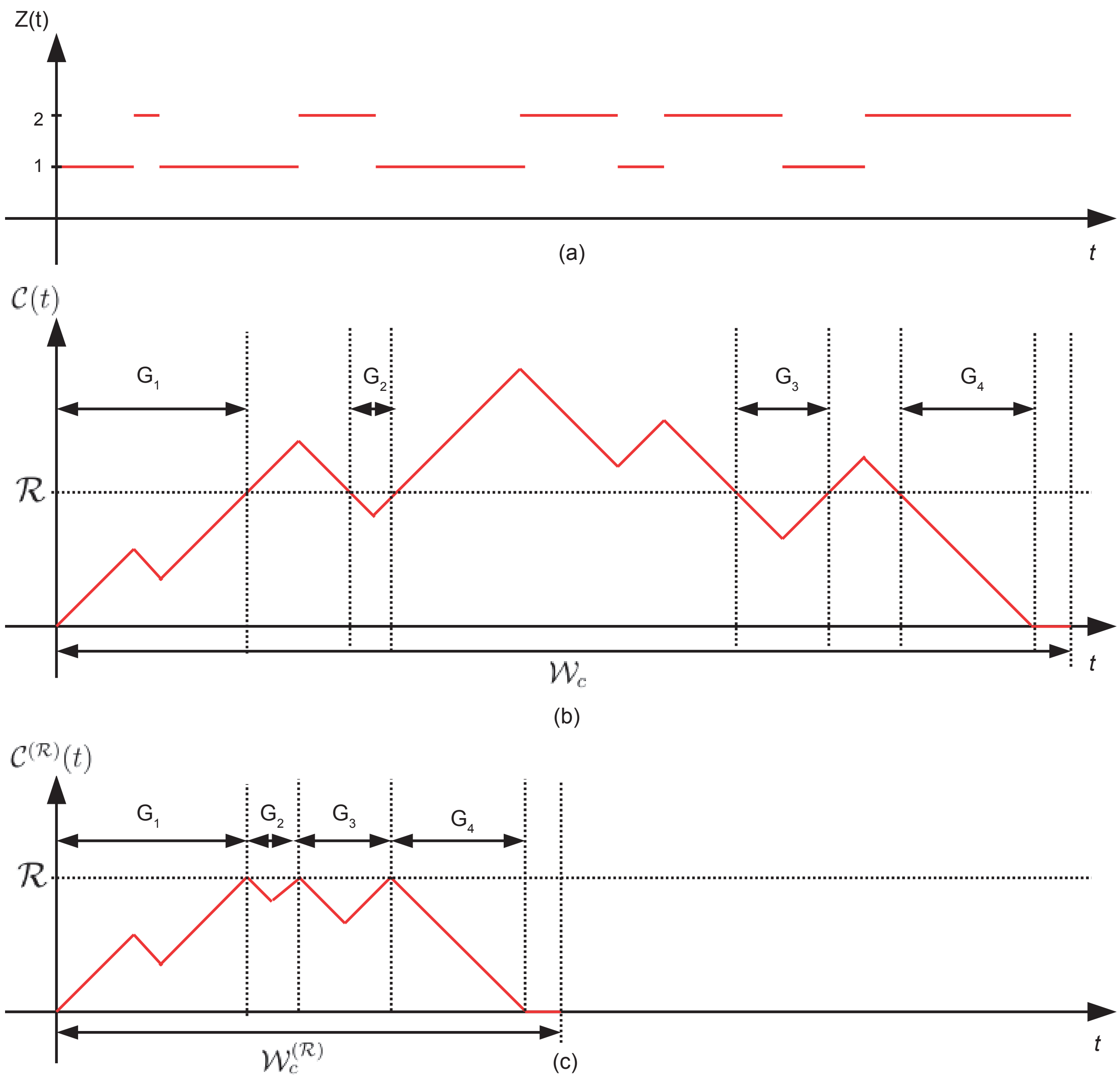

Figure 1.1: A sample path of fluid level and driving process of ordinary fluid queue. (a) $\{Z(t)\}_{t\geq 0}$, (b) $\{\mathcal{C}(t)\}_{t\geq 0}$, (c) $\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\geq 0}$.

the time that the fluid level $\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\geq 0}$ of the finite fluid queue is below level $\mathcal{R}$. Thus, we have

$$\int_0^{\mathcal{W}_c^{(\mathcal{R})}} \mathbb{I}_{(0,\mathcal{R})}\left(\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\geq 0}\right) dt = \int_0^{\mathcal{W}_c} \mathbb{I}_{(0,\mathcal{R})}\left(\{\mathcal{C}(t)\}_{t\geq 0}\right) dt, \tag{1.42}$$

where $\mathbb{I}_{(x,y)}(\bullet)$ is an indicator function. The expression in the left-hand side of (1.42) refers to the duration of time that the sample path of fluid level $\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\geq 0}$ is below $\mathcal{R}$

over $(0, \mathcal{W}_c^{(\mathcal{R})})$, and the expression in the right-hand side of (1.42) refers to the duration of time that the sample path of the fluid level $\{\mathcal{C}(t)\}_{t\geq 0}$ is below $\mathcal{R}$ over $(0, \mathcal{W}_c)$. For instance, as observed in Figure 1.1, both sides of (1.42) equal to $G_1 + G_2 + G_3 + G_4$. Denote by $\mathcal{W}_c$ the complete cycle (the non-empty period + the empty period) of the ordinary fluid queue. The LST of the $\mathcal{W}_c$ is discussed in Section 2.3.3 and the expected value of the $\mathcal{W}_c$ is expressed in (2.18). Denote by $\mathcal{W}_c^{(\mathcal{R})}$ the complete cycle (the non-empty period + the empty period) of the finite fluid queue. By the *Renewal Reward Theorem* [34, Section 7.4, Proposition 7.3], the expected complete cycle of the finite fluid queue can be expressed in terms of the proportion of the expected complete cycle of the ordinary fluid queue, namely

$$\mathbb{E}[\mathcal{W}_c^{(\mathcal{R})}] = \lim_{t\to\infty} \mathbb{P}[\mathcal{C}(t) \leq \mathcal{R}]\mathbb{E}[\mathcal{W}_c] = \mathbb{P}[\mathcal{C} \leq \mathcal{R}]\mathbb{E}[\mathcal{W}_c]. \tag{1.43}$$

Denote by $\pi_{\mathcal{E}}$ the long-run proportion of time that the fluid level of the finite fluid queue is at level 0, by the *Renewal Reward Theorem* ([34, Section 7.4, Popposition 7.3]), we have

$$\pi_{\mathcal{E}} = \frac{1}{\lambda \cdot \mathbb{P}[\mathcal{C} \leq \mathcal{R}] \cdot \mathbb{E}[\mathcal{W}_c]}, \tag{1.44}$$

where λ is the arrival rate of the driving $M/G/1$ queue for the finite fluid queue. Thus, $1/\lambda$ is the expected empty period of the finite fluid queue. By the Smith's Theorem [37], for $0 < x < \mathcal{R}$, the cumulative distribution function of the ordinary fluid queue and finite fluid queue can be expressed in terms of the long-run proportion of the time that the fluid level is below level x, namely

$$\mathbb{P}[\mathcal{C} \leq x] = \frac{\int_0^{\mathcal{W}_c} \mathbb{I}_{(0,\mathcal{R})}\left(\{\mathcal{C}(t)\}_{t\geq 0}\right) dt}{\mathbb{E}[\mathcal{W}_c]}, \tag{1.45}$$

$$\mathbb{P}[\mathcal{C}^{(\mathcal{R})} \leq x] = \frac{\int_0^{\mathcal{W}_c^{(\mathcal{R})}} \mathbb{I}_{(0,\mathcal{R})}\left(\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\geq 0}\right) dt}{\mathbb{E}[\mathcal{W}_c^{(\mathcal{R})}]}. \tag{1.46}$$

The expressions (1.42), (1.45) and (1.46) suggest that

$$\mathbb{P}[\mathcal{C}^{(\mathcal{R})} \le x] = \frac{\mathbb{E}[\mathcal{W}_c] \cdot \mathbb{P}[\mathcal{C} \le x]}{\mathbb{E}[\mathcal{W}_c^{(\mathcal{R})}]}, \quad 0 < x < \mathcal{R}. \tag{1.47}$$

Substituting the expression (1.43) into (1.47) yields

$$\mathbb{P}[\mathcal{C}^{(\mathcal{R})} \le x] = \frac{\mathbb{P}[\mathcal{C} \le x]}{\mathbb{P}[\mathcal{C} \le \mathcal{R}]}, \quad 0 < x < \mathcal{R}. \tag{1.48}$$

Denote by $\widetilde{\mathcal{C}}(s)$ the LST of the pdf of the fluid level of the ordinary fluid queue, then the cdf of the $\mathcal{C}$ can be computed by taking inverse LST of $\widetilde{\mathcal{C}}(s)/s$ using Euler and post-Widder algorithms [1, 2, 3].

1.4.3 Example 3: Distribution of the fluid level in a fluid queue with an upper barrier $\mathcal{R}$

A sample path of the states $\{Z(t)\}_{t\ge0}$ of the driving $M/G/1$ queue and a sample path of the fluid level $\{\mathcal{C}(t)\}_{t\ge0}$ of the fluid queue driven by an $M/G/1$ queue are illustrated in Figure 1.2 (a) and (b) below, respectively. As observed, during the busy periods of the driving queue $\{Z(t) = 1\}_{t\ge0}$, the fluid level increases at rate 1. During the idle period of the driving queue $\{Z(t) = 1\}_{t\ge0}$, the fluid level decreases at rate -1 as long as the fluid level is above 0. We refer the fluid queue driven by an $M/G/1$ queue as an ordinary fluid queue. In this example, we are interested in distribution of the fluid level of a fluid queue with an upper barrier $\mathcal{R}$, such that whenever the fluid level $\{\mathcal{C}(t)\}_{t\ge0}$ of the ordinary fluid queue is above level $\mathcal{R}$, the fluid level $\{\mathcal{C}^{(\mathcal{R})}(t)\}$ of the fluid queue with an upper barrier $\mathcal{R}$ stays at level $\mathcal{R}$. The sample path of the fluid level $\{\mathcal{C}^{(\mathcal{R})}\}_{t\ge0}$ is illustrated in Figure 1.2 (c) below. Here, we assume that the fluid level $\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\ge0}$ of the fluid queue with an upper barrier $\mathcal{R}$ depends on the fluid level in the ordinary fluid queue.

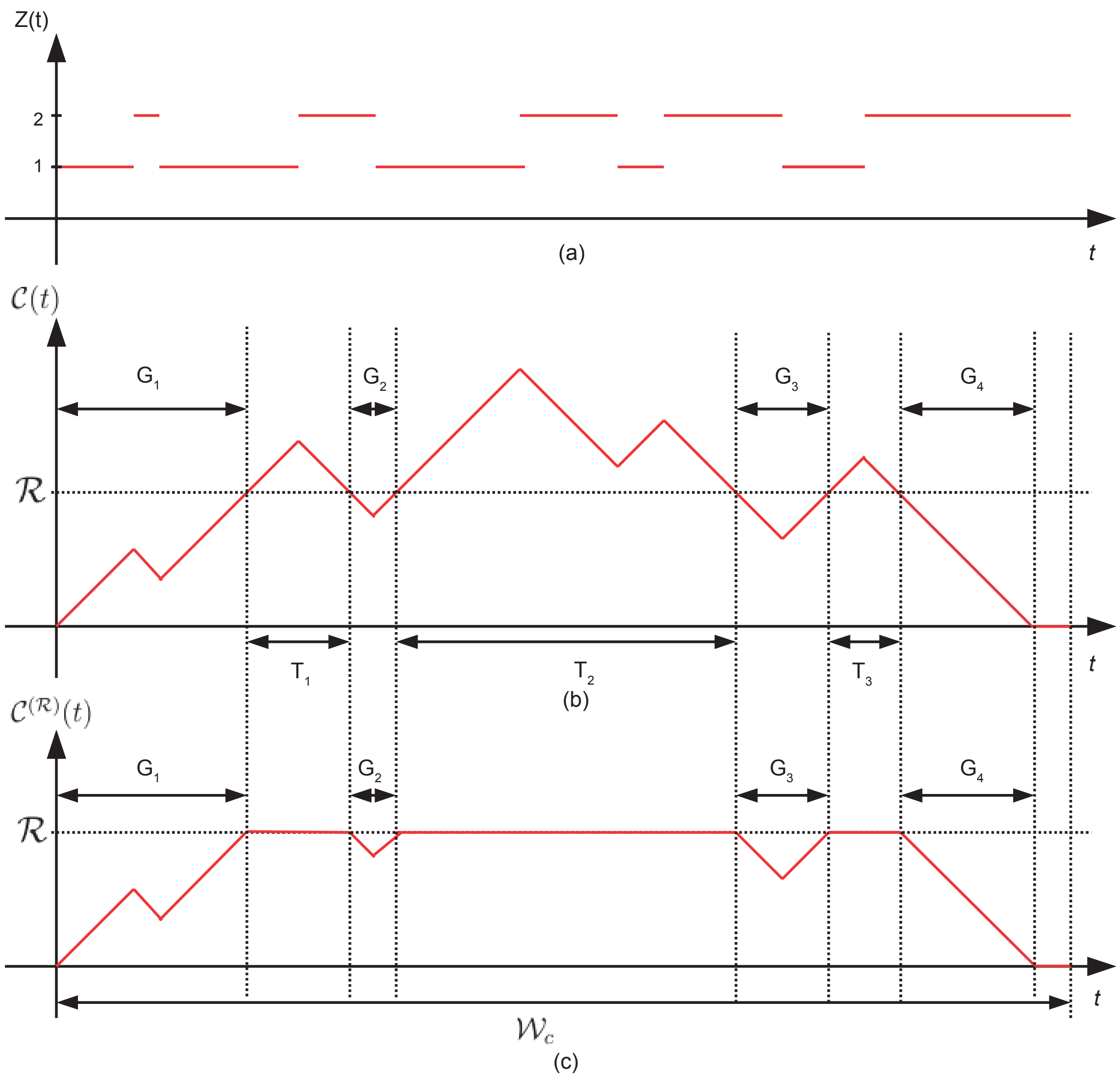

Figure 1.2: The sample paths of the processes: (a) $\{Z(t)\}_{t\geq 0}$, (b) $\{\mathcal{C}(t)\}_{t\geq 0}$, (c) $\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\geq 0}$.

Denote by $\mathcal{W}_c$ the complete cycle (non-empty fluid period plus empty fluid period) of the fluid queue driven by an $M/G/1$ queue discussed in Section 1.2.1. We refer to the fluid queue driven by an $M/G/1$ with arrival rate λ and expected service time $1/\mu$ as ordinary fluid queue. The LST of the pdf of the fluid level $\mathcal{C}$ of the ordinary fluid queue is given in (1.4).

Denote by $\pi_{\mathcal{R}}$ the long-run proportion of time that the fluid level is above level $\mathcal{R}$ and by $\pi_{\mathcal{E}}$ the long-run proportion of time that the fluid level is at level 0. Using the *Renewal*

Reward Theorem ([34, Section 7.4, Popposition 7.3]), we have

$$\pi_{\mathcal{R}} = \frac{\mathbb{E}[\text{time above } \mathcal{R} \text{ in } \mathcal{W}_c]}{\mathbb{E}[\mathcal{W}_c]} = \lim_{t\to\infty} \mathbb{P}[\mathcal{C}(t) \geq \mathcal{R}] = \mathbb{P}[\mathcal{C} \geq \mathcal{R}], \tag{1.49}$$

Letting s go to ∞ in (1.4) yield

$$\pi_{\mathcal{E}} = \frac{1-\lambda\mathbb{E}[B]}{1+\lambda\mathbb{E}[B]} = \frac{\mathbb{E}[I]}{\mathbb{E}[\mathcal{W}_c]}, \tag{1.50}$$

where B and I are the busy and idle period of the driving $M/G/1$ queue of the ordinary fluid queue respectively. Using the expressions (1.49) and (1.50) yields

$$\mathbb{P}[0 < \mathcal{C} < \mathcal{R}] = 1 - \frac{\mathbb{E}[I] + \mathbb{E}[\text{time above } \mathcal{R} \text{ in } \mathcal{W}_c]}{\mathbb{E}[\mathcal{W}_c]} = \mathbb{P}[\mathcal{C} \leq \mathcal{R}] - \pi_{\mathcal{E}}, \tag{1.51}$$

as expected. As observed in Figure 1.2, the long-run proportion of time for $\mathcal{C}^{\mathcal{R}}$ in $(0,\mathcal{R})$ equals to the long-run proportion of time for $\mathcal{C}$ in $(0,\mathcal{R})$, thus we have the duration of the time that the fluid level $\{\mathcal{C}(t)\}_{t\geq 0}$ of the ordinary fluid queue below level $\mathcal{R}$ (i.e. $G_1+G_2+G_3+G_4$) equals the duration of the time that the fluid level $\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\geq 0}$ below $\mathcal{R}$ (i.e. $G_1+G_2+G_3+G_4$). Thus, we have

$$\int_0^{\mathcal{W}_c} \mathbb{I}_{(0,\mathcal{R})}\left(\{\mathcal{C}^{(\mathcal{R})}(t)\}_{t\geq 0}\right) dt = \int_0^{\mathcal{W}_c} \mathbb{I}_{(0,\mathcal{R})}\left(\{\mathcal{C}(t)\}_{t\geq 0}\right) dt. \tag{1.52}$$

Since the expected duration of the complete cycle $\mathbb{E}[\mathcal{W}_c]$ for the ordinary fluid queue and fluid queue with upper barrier $\mathcal{R}$ is the same, by the Smith's Theorem [37], we have

$$\mathbb{P}[0 < \mathcal{C}^{\mathcal{R}} < \mathcal{R}] = \mathbb{P}[0 < \mathcal{C} < \mathcal{R}], \quad 0 < x < \mathcal{R}. \tag{1.53}$$

To summarize (1.49) - (1.53), the cdf of the fluid level of the fluid queue with an upper barrier $\mathcal{R}$ is

$$\mathbb{P}[0 < \mathcal{C}^{\mathcal{R}} < \mathcal{R}] = \mathbb{P}[0 < \mathcal{C} < \mathcal{R}], \quad 0 < x < \mathcal{R}, \tag{1.54}$$

with a point mass at level 0 with probability $\pi_{\mathcal{E}} = \mathbb{P}[\mathcal{C} \leq 0]$ and a point mass at level $\mathcal{R}$ with probability $\pi_{\mathcal{R}} = \mathbb{P}[\mathcal{C} \geq \mathcal{R}]$.

1.5 Conclusion

To summarize this chapter, we introduce an LC method and indicate how to apply it to derive the pdf of fluid level of the finite and infinite-capacity fluid queue driven by an $M/G/1$ queue, and the probability of the busy period of an $M/G/1$ queue being interrupted in a particular model of a finite content fluid queue.

Chapter 2

Infinite capacity fluid system

We consider a fluid queue with infinite capacity driven by an $M/G/1$ queue investigated by J. Virtamo and I. Norros in [44]. As usual, the fluid content empties at rate r_2 *continuously* as long as the content is non-empty, and the content fills at *net* rate r_1 as long as the server of the driving queue is busy. The structure of this chapter is as follows: in Section 2.1, we define the fluid queue driven by an $M/G/1$ queue, and introduce the terminologies used in this dissertation. In Section 2.2, we derive the distribution of the fluid level using the *level crossing* (LC) methods together with a *triangle diagram*; then we formulate the probability density function (pdf) of the fluid level in terms of a Beneš-like series [16, 24, 32]. In Section 2.3, we use a probabilistic interpretation of the Laplace-Stieltjes transform (LST) to interpret the LST of the pdf of the non-empty period of the fluid queue[1] [11, 24, 35]. In Section 2.4, we derive the probability generating functions (pgf) of the number of the *tagged arrivals*, the *arrivals served*, and the *the number of peaks* in a non-empty period of the fluid queue. In Section 2.5, a simulation of the expected values of fluid level and non-empty period is conducted and the simulated results are then compared to the theoretical results. Lastly, we provide two examples to demonstrate the

[1] The Laplace-Stieltjes transform of the non-empty period of the fluid queue was first presented by the Boxma et al. [11] in a different fluid queue model.

connection between the fluid queue and the $M/G/1$ queue with multiple inputs and the $M/G/1$ queue with a balking discipline, in Section 2.6.

2.1 Fluid queue driven by an $M/G/1$ queue

2.1.1 Introduction

Denote by $\{\mathcal{C}(t)\}_{t\geq 0}$ the fluid level at time t, and by $\{Z(t)\}_{t\geq 0}$ the Markov renewal process at time t which takes values in the finite state space $\Omega := \{1, 2, \ldots, n\}$. We refer to the two-dimensional Markov process $\{\mathcal{C}(t), Z(t)\}_{t\geq 0}$ as a fluid queue driven by a Markov renewal process (or more formally, an *environmental process*) $\{Z(t)\}_{t\geq 0}$. The rates at which the fluid content of the fluid queue fills and empties are governed by the process $\{Z(t)\}_{t\geq 0}$ in such a way that

$$\frac{d\mathcal{C}(t)}{dt} = \begin{cases} r_{Z(t)}, & \mathcal{C}(t) > 0, \\ \max(0, r_{Z(t)}), & \mathcal{C}(t) = 0, \end{cases} \tag{2.1}$$

where $r_{Z(t)} \in \mathbb{R}$ is a rate corresponding to the sojourn of $Z(t) = i$. In this chapter, we assume that the Markov renewal process $\{Z(t)\}_{t\geq 0}$ is an $M/G/1$ queue status, which takes values in the finite state space $\Omega = \{1, 2\}$ such that

$$Z(t) = \begin{cases} 1, & \text{if the server of the } M/G/1 \text{ queue is busy,} \\ 2, & \text{if the server of the } M/G/1 \text{ queue is idle,} \end{cases} \tag{2.2}$$

with the *net* input rate $r_1 > 0$ and *continuous* leaking rate $r_2 < 0$.

To recap the characteristics of the fluid queue driven by an $M/G/1$ queue (henceforth fluid queue) discussed above, the fluid content of the fluid queue empties at rate r_2

continuously as long as the content is non-empty, and the content fills at *net* rate r_1 as long as the server of the driving $M/G/1$ queue is busy.

2.1.2 Terminology of the fluid queue

We define

$$\phi(t) = \begin{cases} 0, & \text{if } \mathcal{C}'(t) = 0 \text{ and } Z(t) = 2, \\ 1, & \text{if } \mathcal{C}'(t) > 0 \text{ and } Z(t) = 1, \\ 2, & \text{if } \mathcal{C}'(t) < 0 \text{ and } Z(t) = 2, \end{cases} \tag{2.3}$$

where $\mathcal{C}'(t) := d\mathcal{C}(t)/dt$, as the states of the fluid process $\{\mathcal{C}(t), Z(t)\}_{t\geq 0}$ at time t. We refer to (1) the state $\{\phi(t) = 0\}_{t\geq 0}$ as an ***empty period*** with notation $\mathcal{E}$; (2) the state $\{\phi(t) = 1\}_{t\geq 0}$ as an ***activity period*** with notation $\mathcal{A}$; (3) the state $\{\phi(t) = 2\}_{t\geq 0}$ as a ***silence period*** with notation $\mathcal{S}$. These terminologies have been used in the fluid queue literature [9, 10, 20, 21, 26]. For each complete cycle (non-empty period plus empty period) of the fluid queue, denote by $\mathcal{A}_T$ the ***duration of activity periods***, and by $\mathcal{S}_T$ the ***duration of silence periods*** in a complete cycle of the fluid queue respectively. Finally, denote by $\mathcal{W}_c := \mathcal{A}_T + \mathcal{S}_T + \mathcal{E}$ the ***wet cycle*** (complete cycle) of the fluid queue, and by $\mathcal{W} := \mathcal{A}_T + \mathcal{S}_T$ the ***wet period*** (non-empty period) of the fluid queue. A typical sample path of the fluid level $\{\mathcal{C}(t)\}_{t\geq 0}$, the states of the fluid queue $\{\phi(t)\}_{t\geq 0}$ and the states of the driving $M/G/1$ queue $\{Z(t)\}_{t\geq 0}$ corresponding to the fluid queue are illustrated in Figure 2.1.

Remark 2.1.1. *As observed in Figure 2.1 and using the expressions (2.2) - (2.3), we have* $\{\phi(t) = 1\}_{t\geq 0} = \{Z(t) = 1\}_{t\geq 0}$, *and* $\{\phi(t) = 0\}_{t\geq 0} \bigcup \{\phi(t) = 2\}_{t\geq 0} = \{Z(t) = 2\}_{t\geq 0}$. *These properties are important when finding the long-run proportion of time of the empty, activity, and silence periods.*

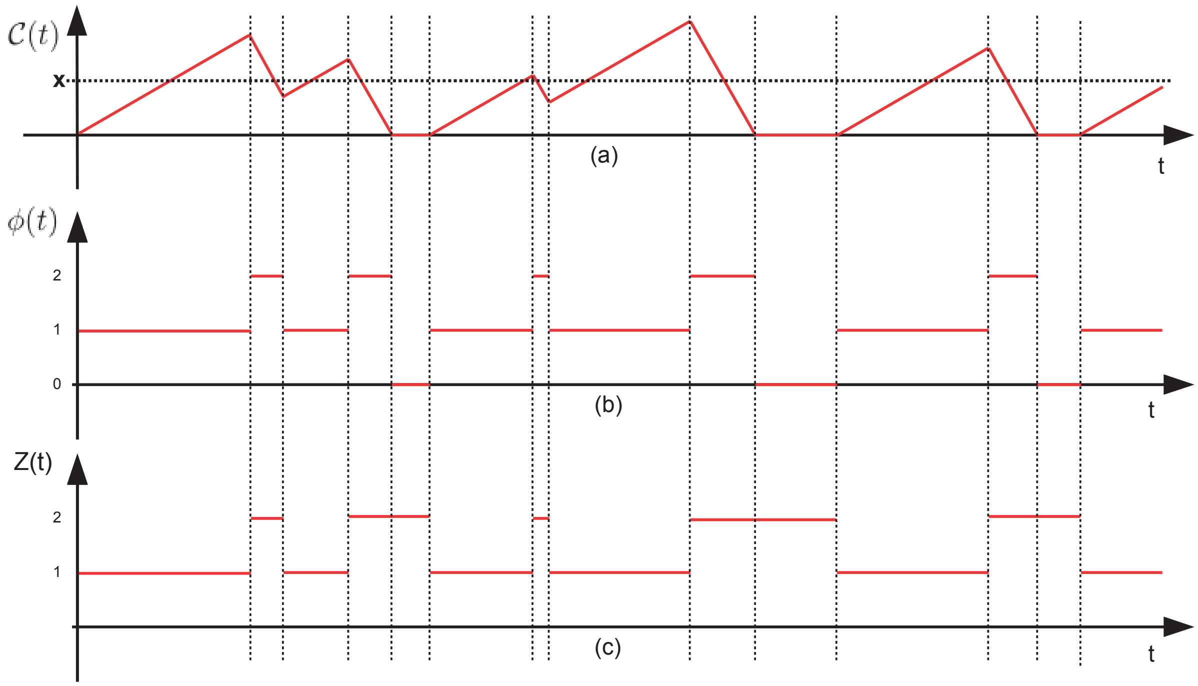

Figure 2.1: (a) A typical sample path of fluid level. (b) The states of the fluid queue. (c) The states of the driving $M/G/1$ queue.

2.1.3 Definition of the probability density function of the fluid level

In this section, we define a partial probability density function (ppdf) and a total (or marginal) pdf of the fluid level. For $x > 0$, define the steady-state partial cumulative probability distribution function (cdf) of the fluid level as

$$F_i(x) := \lim_{t\to\infty} \mathbb{P}[0 < \mathcal{C}(t) \leq x, \phi(t) = i] + \lim_{t\to\infty} \mathbb{P}[\phi(t) = 0], \quad i = 1,\ 2, \tag{2.4}$$

where $\phi(t)$ is defined in (2.3). The steady-state partial pdf is defined as

$$f_i(x) = \frac{d}{dx}\left(\lim_{t\to\infty} \mathbb{P}\left[\mathcal{C}(t) \leq x, \phi(t) = i\right]\right), \quad i = 1,\ 2. \tag{2.5}$$

Define the total pdf of the fluid level as

$$f(x) = \frac{d}{dx}\left(\lim_{t\to\infty} \mathbb{P}[\mathcal{C}(t) \le x]\right), \quad x > 0. \tag{2.6}$$

The total pdf of the fluid level is

$$f(x) = f_1(x) + f_2(x), \quad x > 0. \tag{2.7}$$

We remark that the partial pdf $f_1(x)$ corresponds to the fluid level in the activity period, and the partial pdf $f_2(x)$ corresponds to the fluid level in the silence period. More importantly, the expression (2.4) indicates that there has a point mass for the distribution of the fluid level at 0. Finally, it is important to highlight that

$$\int_0^\infty f_1(x)dx \neq 1 \quad \text{and} \quad \int_0^\infty f_2(x)dx \neq 1. \tag{2.8}$$

2.1.4 Stability condition of the fluid queue

For a fixed $x > 0$ and $t > 0$, define $U_t(x)$ and $D_t(x)$ be the number of up- and downcrossings of level x during the time interval $(0, t)$ respectively. Brill [12, Section 6.2.7, pp. 304 - 309] shows that the up- and downcrossing rate of level x during the time interval $(0, t)$ are

$$\lim_{t\to\ \infty} \frac{U_t(x)}{t} \overset{a.s.}{=} r_1 f_1(x) \quad \text{and} \quad \lim_{t\to\ \infty} \frac{D_t(x)}{t} \overset{a.s.}{=} r_2 f_2(x), \quad x > 0, \tag{2.9}$$

where $f_1(x)$ and $f_2(x)$ are defined in (2.5) respectively. Here r_1 and r_2 are the *net* input rate and the *continuous* leaking rate of the fluid queue respectively. Let $\pi_{\mathcal{E}} := \lim_{t\to\infty} \mathbb{P}[\phi(t) = 0]$ be the point mass of the fluid level at 0. Using a LC argument, the rate at which the sample path of the fluid level enters and leaves level 0 are equal. Hence, by (2.9), we have $r_1 f_1(0^+) = \lambda\pi_{\mathcal{E}} = r_2 f_2(0^+)$. The term $\lambda\pi_{\mathcal{E}}$ is the rate at which an arrival arrives to the empty $M/G/1$ queue while the fluid content is also empty, and this particular arrival initiates a busy period in the driving $M/G/1$ queue and causes the sample path to leave level 0. Each time point when $\{\mathcal{C}(t)\}_{t\geq 0}$ leaves level 0 is a regenerative point [36, 43]. Hence, $\{\mathcal{C}(t)\}_{t\geq 0}$ is a regenerative process with wet cycle $\mathcal{W}_c$ that forms a renewal process. By the *Elementary Renewal Theorem* [34, Section 7.3, pp. 407 - 416], $\mathbb{E}[\mathcal{W}_c] = 1/(\lambda\pi_{\mathcal{E}})$. By the *Renewal Reward Theorem* [34, Proposition 7.3, pp. 416 - 417], the long-run proportion of time that $\{\phi(t) = 1 \text{ or } 2\}_{t\geq 0}$ is

$$\frac{\mathbb{E}[\{\phi(t) = 1 \text{ or } 2\}_{t\geq 0}]}{\mathbb{E}[\{\phi(t) = 0\}_{t\geq 0}] + \mathbb{E}[\{\phi(t) = 1 \text{ or } 2\}_{t\geq 0}]} = \frac{\mathbb{E}[\mathcal{W}]}{\mathbb{E}[\mathcal{W}_c]} = 1 - \pi_{\mathcal{E}}. \tag{2.10}$$

For each wet cycle, the amount of fluid entering equals the amount of fluid leaving the fluid content. Consequently, we have

$$r_1 \lim_{t\to\infty} \mathbb{P}[\{\phi(t) = 1\}_{t\geq 0}] = r_2 \lim_{t\to\infty} \mathbb{P}[\{\phi(t) = 2\}_{t\geq 0}]. \tag{2.11}$$

The expression (2.11) leads to

$$r_1 \mathcal{A}_T = r_2 \mathcal{S}_T, \tag{2.12}$$

given that $\mathbb{E}[\mathcal{W}_c] < \infty$. In addition, by the *Remark 2.1.1*, we have $\forall t > 0, \mathbb{P}[\phi(t) = 1] = \mathbb{P}[Z(t) = 1]$, i.e., $\mathcal{A}$ and B are identical distributed, where B is the busy period of the driving $M/G/1$ queue. Since each wet cycle consists multiple of busy cycles ($\mathcal{N}_p$) of the driving $M/G/1$ queue, and $\mathbb{E}[\mathcal{A}_T] = \mathbb{E}[\mathcal{N}_p]\cdot\mathbb{E}[\mathcal{A}] = \mathbb{E}[\mathcal{N}_p]\cdot\mathbb{E}[B]$, we have the long-run proportion of the time of the $\mathcal{A}_T$ equals the traffic intensity of the driving $M/G/1$ queue, namely: $\rho := \lambda/\mu := (\mathbb{E}[\mathcal{N}_p]\cdot\mathbb{E}[B]) / (\mathbb{E}[\mathcal{N}_p]\cdot(\mathbb{E}[B] + \mathbb{E}[I]))$, where I is the idle period of the driving $M/G/1$ queue with arrival rate λ and expected service time $1/\mu$. Using this property with (2.10) and (2.11), we obtain

$$\pi_{\mathcal{E}} = 1 - \left(1 + \frac{r_1}{r_2}\right)\rho = 1 - \left(1 + \frac{r_1}{r_2}\right)\left(\frac{\mathbb{E}[B]}{\mathbb{E}[B] + \mathbb{E}[I]}\right). \tag{2.13}$$

Setting $0 < \pi_{\mathcal{E}} \leq 1$ yields the stability condition of the fluid queue, namely

$$r_2\mathbb{E}[I] > r_1\mathbb{E}[B]. \tag{2.14}$$

If this condition is not satisfied, then the steady-state distribution of the fluid level does not exist. Considering that the activity periods of the fluid queue are governed by the busy periods of the $M/G/1$ queue, the stability condition (2.14) states that the expected (*net*) fluid entering the fluid content during the busy period of the $M/G/1$ queue has to be less than the expected fluid leaving the fluid content during the idle period of the $M/G/1$ queue.

2.1.5 Expected values of the fluid queue

Denote by $\mathcal{N}_p$ the number of complete busy cycles of the $M/G/1$ queue in a wet cycle. Figure 2.2 illustrates the composition of a wet cycle of the fluid queue in terms of activity and silence periods, and empty periods.

On one hand, the expected wet cycle can be written as

$$\mathbb{E}[\mathcal{W}_c] = \mathbb{E}\left[\sum_{i=1}^{\mathcal{N}_p}(B_i + I_i)\right] = \mathbb{E}[\mathcal{N}_p]\left(\mathbb{E}[B] + \mathbb{E}[I]\right). \tag{2.15}$$

On the other hand, since the arrival process of the driving $M/G/1$ queue is a Poisson process, by the memoryless property and (2.12), the expected wet cycle can be written as

$$\mathbb{E}[\mathcal{W}_c] = \mathbb{E}[\mathcal{A}_T] + \mathbb{E}[\mathcal{S}_T] + \mathbb{E}[\mathcal{E}] = \left(1 + \frac{r_1}{r_2}\right)\mathbb{E}[\mathcal{N}_p]\mathbb{E}[B] + \mathbb{E}[I], \tag{2.16}$$

where $\mathbb{E}[\mathcal{E}] = \mathbb{E}[I]$, and $\mathcal{A}_T, \mathcal{S}_T$, and $\mathcal{E}$ are defined in Section 2.1.1.

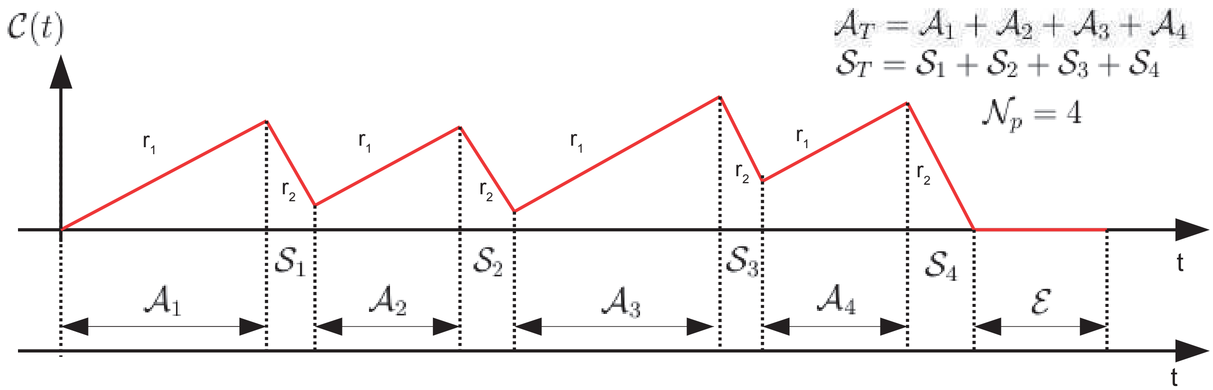

Figure 2.2: Composition of a single wet cycle

Equating (2.15) and (2.16), and collecting the terms yield

$$\mathbb{E}[\mathcal{N}_p] = \frac{\mathbb{E}[I]}{\mathbb{E}[I] - \frac{r_1}{r_2}\mathbb{E}[B]}. \tag{2.17}$$

The expression (2.17) implies, by (2.12) and (2.16), that

$$\mathbb{E}[\mathcal{A}_T] = \frac{\mathbb{E}[I]\mathbb{E}[B]}{\mathbb{E}[I] - \frac{r_1}{r_2}\mathbb{E}[B]} \quad \text{and} \quad \mathbb{E}[\mathcal{W}] = \frac{\mathbb{E}[B](\mathbb{E}[I])^2}{\mathbb{E}[I] - \frac{r_1}{r_2}\mathbb{E}[B]}. \tag{2.18}$$

As observed in Figure 2.2, it is important to highlight that $\mathcal{N}_p$ can be also interpreted as the number of peaks in a wet period.

2.2 Probability distribution of the fluid level

2.2.1 Downcrossing and upcrossing rates of level x

Recall that, for $x > 0$ and $t > 0$, the upcrossing and downcrossing rates at which the sample path of the fluid level crosses level x from above and below are given in (2.9). In addition, the rate at which the sample path of the fluid level upcrosses level x can be expressed as[2]

$$r_1 f_1(x) = \lambda \pi_{\mathcal{E}} \overline{B}\left(\frac{x}{r_1}\right) + \lambda \int_{y=0^+}^{y=x} \overline{B}\left(\frac{x-y}{r_1}\right) f_2(y)dy, \tag{2.19}$$

where $\overline{B}(\bullet)$ is the complementary cumulative distribution function (ccdf) of the busy period of the driving $M/G/1$ queue. In addition, since the upcrossing rate of level x equals the downcrossing rate, the expression (2.19) can be interpreted as the downcrossing rate of level x (i.e., $r_2 f_2(x)$) as well. The first term on the right hand side of (2.19) is the rate at which the sample path upcrosses level x starting at level 0, and the second term of the right hand side of (2.19) is the rate at which the sample path upcrosses level of x starting from level y in the interval $(0, x)$. For details, readers are referred to [12, Section 1.6 and 1.7, pp. 13 - 17].

Observation 2.2.1. *Letting $r_1 = r_2 = 1$ in (2.19) yields a LC equation analogous to the equation for the pdf of the virtual waiting time for an $M/G/1$ queue (see [12, p. 17] for the details). It hints that the LC equation of the fluid level links to the LC equation of the virtual waiting time of an $M/G/1$ queue with adjustment for the rates. However, it is important to point out that $\pi_{\mathcal{E}}$ is the point mass at level 0 for the fluid level instead of the point mass of the virtual waiting time at level 0 for an $M/G/1$ queue.*

[2] $r_1 f_1(x)$ is the upcrossing rate of level x, and $r_2 f_2(x)$ is the downcrossing rate of level x.

2.2.2 Triangle diagram

In this section, we introduce a *triangle diagram* for the sample path of the fluid level which can be applied to express the upcrossing rate $r_1 f_1(x)$ and the downcrossing rate $r_2 f_2(x)$ in terms of the pdf of the fluid level $f(x)$.

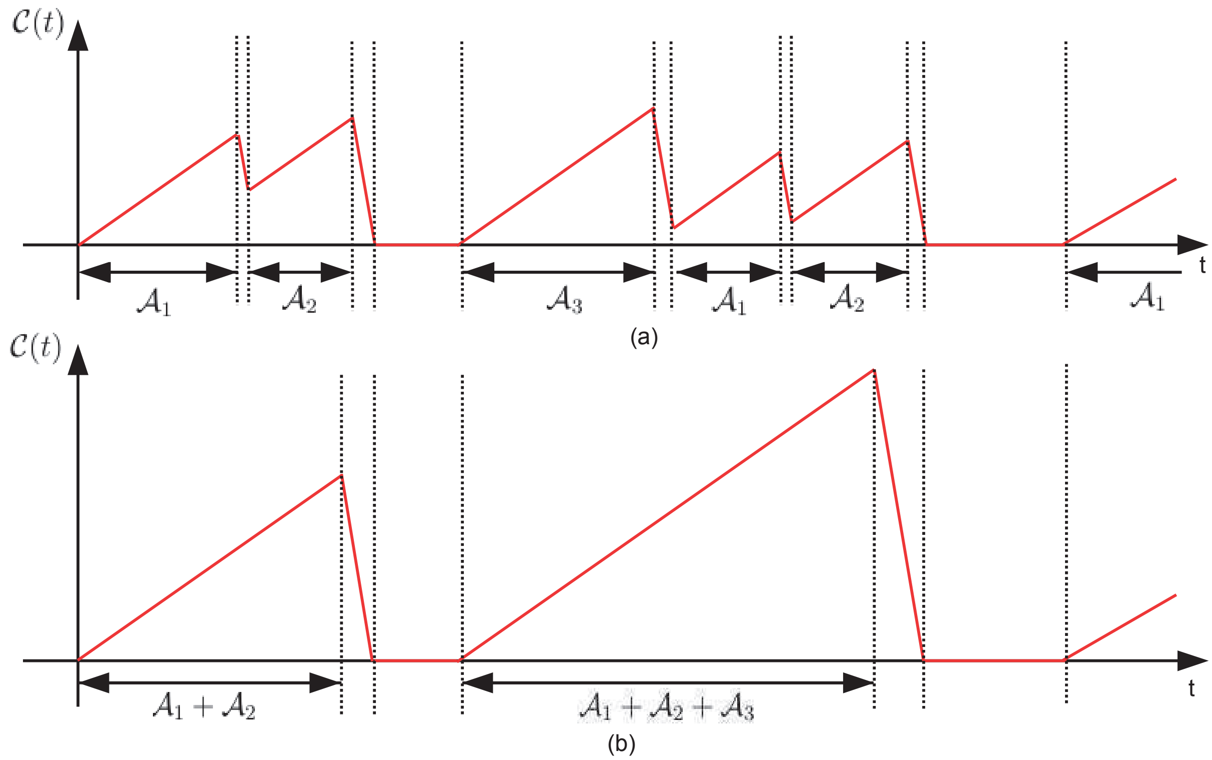

Figure 2.3: (a) A typical sample path of fluid level. (b) A corresponding sample path of fluid level.

Figure 2.3 illustrates the sample paths of the fluid level. In Figure 2.3, both sample paths encompass two complete wet cycles and one incomplete wet cycle. Specifically, in Figure 2.3 (a), the first wet cycle encompasses two activity and silence periods, and the second wet cycle encompasses three activity and silence periods. In Figure 2.3 (b), both wet cycles encompass one activity and one silence periods. Although, the fluid level at time t in Figure 2.3 (a) and (b) are different in general, the duration of the activity $\mathcal{A}_T$ and silence $\mathcal{S}_T$ periods, and wet cycles $\mathcal{W}_c$ are equal. This suggests that relocating the activity and silence periods in a wet cycle do not affect the duration of activity and silence periods; or the duration of wet period. Thus, if we would like to express $f_1(x)$ and $f_2(x)$ in terms of $f(x)$, then the information achieved from 2.3 (b) are identical to the information achieved from 2.3 (a). Hence, to simplify our work, in this section, we focus on Figure 2.3 (b). It is important to point out here that the processes demonstrated in Figure 2.3 (a) and (b) are both regenerative processes. Thus by the theory of regenerative processes, for these particular figures, the long-run proportion of time of the activity periods achieved using the first wet cycle equals the long-run proportion of time of the activity periods achieved using the second wet cycle. Thus, we will express $f_1(x)$ and $f_2(x)$ in terms of $f(x)$ using the first wet period, and refer to the diagram of as a *triangle diagram*. Figure 2.4 illustrates the *triangle diagram* of the fluid level in a wet cycle.

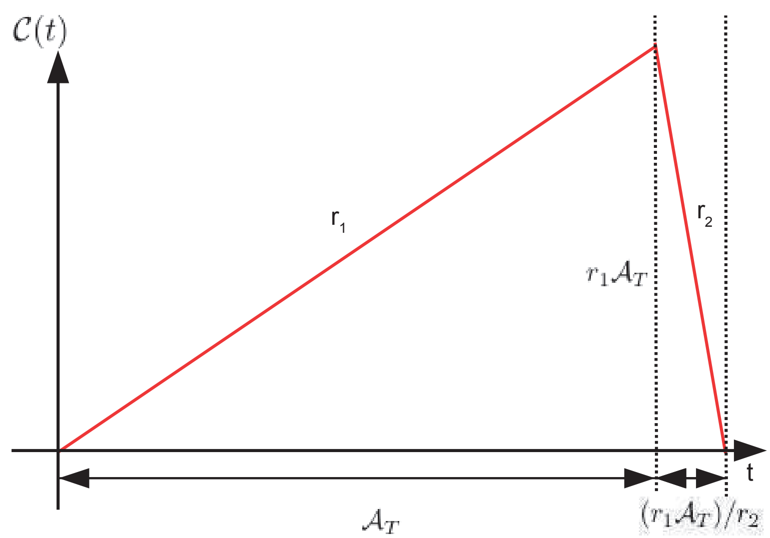

Figure 2.4: Triangle diagram of the fluid level

Denote by $\mathcal{A}_T = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3$ the duration of the activity periods in the first wet cycle in Figure 2.4; then the duration of the silence periods is $(r_1\mathcal{A}_T)/r_2$. Thus, the proportions of time of the activity and silence periods are

$$\frac{r_2}{r_1 + r_2} \quad \text{and} \quad \frac{r_1}{r_1 + r_2}, \tag{2.20}$$

respectively. The above expression implies that the weights associated with $f(x)$ on $\{\phi(t) = 1\}_{t \geq 0}$ and $\{\phi(t) = 2\}_{t \geq 0}$ are $r_2/(r_1 + r_2)$ and $r_1/(r_1 + r_2)$ respectively. These properties suggest that ($x > 0$)

$$f_1(x) = \frac{r_2}{r_1 + r_2} f(x) \quad \text{and} \quad f_2(x) = \frac{r_1}{r_1 + r_2} f(x). \tag{2.21}$$

The expression (2.21) does not violate the principle of rate balance condition (see [12, p. 16] for the details) at which the necessary condition for (2.21) to be true is

$$\lim_{t \to \infty} \frac{D_t(x)}{t} = r_1 f_1(x) = \frac{r_1 r_2}{r_1 + r_2} f(x) = r_2 f_2(x) = \frac{r_1 r_2}{r_1 + r_2} f(x) = \lim_{t \to \infty} \frac{U_t(x)}{t} \tag{2.22}$$

(i.e., the downcrossing rate of level x equals the upcrossing rate of level x.) Finally, it is important to highlight that solving a system of linear equations (i.e., (2.7) and (2.22) together) gives the same conclusion as shown in (2.21). The intention in this section is to provide an alternative way to express the rate at which the sample path upcrosses and downcrosses level x in terms of the pdf of the fluid level as well as visualizing the concepts behind the mathematics.

We end this section by highlighting the limitations of the *triangle diagram* when applied to express $f_1(x)$ and $f_2(x)$ in terms of $f(x)$. The *triangle diagram* approach discussed in this section relies on the two assumptions: (1) the *net* input and *continuous* leaking rate are constant, and (2) the wet cycle is a regenerative process.

2.2.3 Level-crossing equations

Substituting (2.21) into (2.19) yields an LC equation for the pdf of the fluid level, namely

$$\frac{r_1 r_2}{r_1 + r_2} f(x) = \lambda \pi_{\mathcal{E}} \overline{B}\left(\frac{x}{r_1}\right) + \lambda \left(\frac{r_1}{r_1 + r_2}\right) \int_{y=0^+}^{y=x} \overline{B}\left(\frac{x-y}{r_1}\right) f(y) dy, \quad x > 0. \tag{2.23}$$

Remark 2.2.2. *The expression (2.23) can be interpreted in two ways: (1) equating two different expressions of the upcrossing rate at level x. (2) balancing upcrossing and downcrossing rates of level x, since $r_1 f_1(x) = r_2 f_2(x)$.*

2.2.4 Beneš-like series for PDF of fluid level

In this section, we formulate the pdf of the fluid level using a LC argument (see [13] for more details) which explains the meaning of Beneš' mathematical series first introduced by Beneš in [6] for the pdf of the virtual waiting time of an $M/G/1$ queue (discussed and made noteworthy by Kleinrock [24]). Two completely different explanations of the Beneš' series for the pdf of the virtual waiting time of an $M/G/1$ queue are given by Cooper and Niu in [16] and Prabhu in [32].

Consider the equation (2.23). Applying a technique similar to that in Brill [13], we multiply by $r_1 + r_2$ and divide by $r_1 r_2$ to yield

$$f(x) = \lambda \pi_{\mathcal{E}} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \overline{B} \left(\frac{x}{r_1} \right) + \frac{\lambda}{r_2} \int_{y=0^+}^{y=x} \overline{B} \left(\frac{x-y}{r_1} \right) f(y) dy, \quad x > 0. \tag{2.24}$$

We divide and multiply by $\mathbb{E}[B]$ in the first term of the right-hand side of above equation to yield

$$\lambda \mathbb{E}[B] \pi_{\mathcal{E}} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \frac{1}{\mathbb{E}[B]} \overline{B} \left(\frac{x}{r_1} \right) = \rho_B \pi_{\mathcal{E}} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) g^{*1} \left(\frac{x}{r_1} \right), \quad x > 0, \tag{2.25}$$

where $\rho_B = \lambda \mathbb{E}[B]$ and $g^{*k}(x/r_1)$ is the k-fold self convolution of the residual time of the busy period at x/r_1 of the driving $M/G/1$ queue, $k = 1, 2, \ldots$. Next, we substitute the complete expression for $f(y)$ obtained from (2.24) back into the integral part of (2.24).

This gives (for $x > 0$)

$$\begin{aligned}
\frac{\lambda}{r_2}\int_{y=0}^{y=x}\overline{B}\left(\frac{x-y}{r_1}\right)f(y)dy
&= \frac{\lambda}{r_2}\int_{y=0}^{y=x}\overline{B}\left(\frac{x-y}{r_1}\right)\left[\lambda\pi_{\mathcal{E}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\overline{B}\left(\frac{y}{r_1}\right)\right]dy \\
&+ \frac{\lambda}{r_2}\int_{y=0}^{y=x}\overline{B}\left(\frac{x-y}{r_1}\right)\left[\frac{\lambda}{r_2}\int_{z=0}^{z=y}\overline{B}\left(\frac{y-z}{r_1}\right)f(z)dz\right]dy \\
&= \frac{\lambda^2}{r_2}\pi_{\mathcal{E}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\int_{y=0}^{y=x}\overline{B}\left(\frac{x-y}{r_1}\right)\overline{B}\left(\frac{y}{r_1}\right)dy \\
&+ \left(\frac{\lambda}{r_2}\right)^2\int_{y=0}^{y=x}\int_{z=0}^{z=y}\overline{B}\left(\frac{x-y}{r_1}\right)\overline{B}\left(\frac{y-z}{r_1}\right)f(z)dzdy. \qquad (2.26)
\end{aligned}$$

In the second-last term of (2.26), multiplying and dividing $(\mathbb{E}[B])^2$ yields

$$\begin{aligned}
\frac{\lambda^2}{r_2}\pi_{\mathcal{E}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\int_{y=0}^{y=x}\overline{B}\left(\frac{x-y}{r_1}\right)\overline{B}\left(\frac{y}{r_1}\right)dy \\
= \frac{\rho_B^2}{r_2}\pi_{\mathcal{E}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\int_{y=0}^{y=x}\frac{1}{\mathbb{E}[B]}\overline{B}\left(\frac{x-y}{r_1}\right)\frac{1}{\mathbb{E}[B]}\overline{B}\left(\frac{y}{r_1}\right)dy. \qquad (2.27)
\end{aligned}$$

Notice that the integral from equation (2.27) can be further simplified to

$$\begin{aligned}
\int_{y=0}^{y=x}\frac{1}{\mathbb{E}[B]}\overline{B}\left(\frac{x-y}{r_1}\right)\frac{1}{\mathbb{E}[B]}\overline{B}\left(\frac{y}{r_1}\right)dy &= r_1\int_{u=0}^{u=x/r_1}\frac{1}{\mathbb{E}[B]}\overline{B}\left(\frac{x}{r_1}-u\right)\frac{1}{\mathbb{E}[B]}\overline{B}(u)du \\
&= r_1 g^{*2}\left(\frac{x}{r_1}\right). \qquad (2.28)
\end{aligned}$$

Thus equation (2.27) can be further simplified to

$$\begin{aligned}
\frac{\lambda^2}{r_2}\pi_{\mathcal{E}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\int_{y=0}^{y=x}\overline{B}\left(\frac{x-y}{r_1}\right)\overline{B}\left(\frac{y}{r_1}\right)dy \\
&= \frac{r_1}{r_2}\rho_B^2\pi_{\mathcal{E}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)g^{*2}\left(\frac{x}{r_1}\right) \\
&= \pi_{\mathcal{E}}\left(\frac{r_1+r_2}{r_1^2}\right)\left(\frac{r_1}{r_2}\rho_B\right)^2 g^{*2}\left(\frac{x}{r_1}\right). \qquad (2.29)
\end{aligned}$$

Similarly, substituting $f(\bullet)$ from equation (2.24) into the last term of equation (2.26) yields (for $x > 0$)

$$
\begin{aligned}
&\left(\frac{\lambda}{r_2}\right)^2 \int_{y=0}^{y=x} \int_{z=0}^{z=y} \overline{B}\left(\frac{x-y}{r_1}\right) \overline{B}\left(\frac{y-z}{r_1}\right) f(z) dz dy \\
&= \left(\frac{\lambda}{r_2}\right)^2 \int_{y=0}^{y=x} \int_{z=0}^{z=y} \overline{B}\left(\frac{x-y}{r_1}\right) \overline{B}\left(\frac{y-z}{r_1}\right) \left[\lambda \pi_{\mathcal{E}} \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \overline{B}\left(\frac{z}{r_1}\right)\right] dz dy + \ldots \\
&= \left(\frac{\lambda^3}{r_2^2}\right) \pi_{\mathcal{E}} \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \int_{y=0}^{y=x} \int_{z=0}^{z=y} \overline{B}\left(\frac{x-y}{r_1}\right) \overline{B}\left(\frac{y-z}{r_1}\right) \overline{B}\left(\frac{z}{r_1}\right) dz dy + \ldots .
\end{aligned}
$$

Multiplying and dividing by $(E[B])^3$ in the above equation yields

$$
\begin{aligned}
&\left(\frac{\lambda}{r_2}\right)^2 \int_{y=0}^{y=x} \int_{z=0}^{z=y} \overline{B}\left(\frac{x-y}{r_1}\right) \overline{B}\left(\frac{y-z}{r_1}\right) f(z) dz dy \\
&\qquad = \pi_{\mathcal{E}} \left(\frac{r_1 + r_2}{r_1^2}\right) \left(\frac{r_1}{r_2} \rho_B\right)^3 g^{*3} \left(\frac{x}{r_1}\right) + \ldots . \qquad (2.30)
\end{aligned}
$$

Repeatedly substituting for $f(\bullet)$, we get

$$
f(x) = \begin{cases} \pi_{\mathcal{E}}, & x = 0, \\ \pi_{\mathcal{E}} \left(\frac{r_1 + r_2}{r_1^2}\right) \sum_{k=1}^{\infty} \left(\frac{r_1}{r_2} \rho_B\right)^k g^{*k} \left(\frac{x}{r_1}\right), & x > 0. \end{cases} \qquad (2.31)
$$

The expression (2.31) suggests that for $x > 0$, $f(x)$ is the weighted sum of convolved residual busy periods of B ar x/r_1. Integrating (2.31) with respect to x over $(0^+, \infty)$ yields

$$
\begin{aligned}
\int_{x=0^+}^{x=\infty} f(x) dx &= \int_{x=0^+}^{x=\infty} \pi_{\mathcal{E}} \left(\frac{r_1 + r_2}{r_1^2}\right) \sum_{k=1}^{\infty} \left(\frac{r_1}{r_2} \rho_B\right)^k g^{*k} \left(\frac{x}{r_1}\right) dx \\
&= \pi_{\mathcal{E}} \left(\frac{r_1 + r_2}{r_1}\right) \sum_{k=1}^{\infty} \left(\frac{r_1}{r_2} \rho_B\right)^k . \qquad (2.32)
\end{aligned}
$$

Using the normalization condition,

$$
\pi_{\mathcal{E}} + \int_{x=0^+}^{x=\infty} f(x) dx = 1, \qquad (2.33)
$$

and (2.32) gives

$$\begin{aligned}
\pi_{\mathcal{E}} &= \left(1 + \left(\frac{r_1 + r_2}{r_1}\right) \sum_{k=1}^{\infty} \left(\frac{r_1}{r_2}\rho_B\right)^k\right)^{-1} \\
&= \left(1 + \left(\frac{r_1 + r_2}{r_1}\right) \frac{r_1}{r_2}\lambda\mathbb{E}[B]\frac{1}{1 - \frac{r_1}{r_2}\lambda\mathbb{E}[B]}\right)^{-1} \\
&= \frac{r_2 - r_1\lambda\mathbb{E}[B]}{r_2 + r_2\lambda\mathbb{E}[B]} \qquad (2.34)
\end{aligned}$$

where we assume that the geometric series in equation (2.32) converges to a number. Letting $0 < \pi_{\mathcal{E}} < 1$, we obtain

$$\frac{r_1}{r_2}\rho_B < 1 \iff r_1\mathbb{E}[B] < r_2\mathbb{E}[I] \qquad (2.35)$$

which is the stability condition of the fluid queue in (2.14). Finally, since B is the busy period of the driving $M/G/1$ queue with arrival rate λ and expected service time $1/\mu$, substituting $\mathbb{E}[B] = 1/(\mu - \lambda)$ into (2.34) gives (2.13).

2.3 Laplace-Stieltjes transform

2.3.1 Introduction

Denote by X the random variable that has a legitimate probability density function $f(x),\ x \geq 0$. The Laplace-Stieltjes transform (LST) of $f(x)$ named after the mathematician Pierre Simon Laplace, is defined as

$$\mathcal{L}_f(s) = \int_{0^-}^{\infty} e^{-sx} dF(x), \qquad (2.36)$$

where $F(x)$ is the cdf of X. The equation (2.36) is a mathematical transformation which at this point has no meaning attached. However,there is a probabilistic interpretation of $\mathcal{L}(s)$. One can introduce a new random variable Y which is exponentially distributed with rate $s > 0$ and refer to Y as a catastrophe random variable. Using this, we have

$$\widetilde{F}(s) = \int_{0^-}^{\infty} \mathbb{P}[Y > x] dF(x) = \mathbb{P}[X < Y], \tag{2.37}$$

where $\widetilde{F}(s)$ is the LST of $f(x)$. This suggests that the LST of the pdf $f(x)$ can be interpreted as the probability that the event associated with X occurs before the event (catastrophe) associated with Y. A complete discussion of catastrophe processes can be found in Kleinrock [24, pp. 267 - 269], Nelson [31, pp. 197 - 204], and Roy et al. [35]

2.3.2 Laplace-Stieltjes transform of the fluid level

We have expressed the pdf of the fluid level previously in Section 2.2.4 as a Beneš-like series. In this section, we derive the LST of the pdf of the fluid level. Denote by $\mathcal{C}$ the steady-state random variable of the fluid level and by B the steady-state random variable of the busy period of the driving $M/G/1$ queue. For $s > 0$, denote by

$$\widetilde{\mathcal{C}}(s) = \pi_{\mathcal{E}} + \int_{0^-}^{\infty} e^{-sx} d\mathbb{P}[\mathcal{C} < x] \quad \text{and} \quad \widetilde{B}(s) = \int_{0^-}^{\infty} e^{-sx} d\mathbb{P}[B < x] \tag{2.38}$$

the LST of the pdf of the fluid level and the pdf of the busy period of the driving $M/G/1$ queue respectively. Multiplying e^{-sx} to both sides of (2.23) and integrating with respect to x over $(0, \infty)$ yield

$$\begin{aligned} r_1 r_2 s \left(\widetilde{\mathcal{C}}(s) - \pi_{\mathcal{E}}\right) &= (r_1 + r_2)\lambda\pi_{\mathcal{E}} \left(1 - \widetilde{B}(r_1 s)\right) \\ &\quad + r_1\lambda \left(1 - \widetilde{B}(r_1 s)\right)\left(\widetilde{\mathcal{C}}(s) - \pi_{\mathcal{E}}\right), \quad s > 0. \end{aligned} \tag{2.39}$$

Collecting the terms yields

$$\widetilde{\mathcal{C}}(s) = \pi_{\mathcal{E}} \left(\frac{r_1 r_2 s + r_2 \lambda (1 - \widetilde{B}(r_1 s))}{r_1 r_2 s - r_1 \lambda (1 - \widetilde{B}(r_1 s))} \right), \quad s > 0. \tag{2.40}$$

Using the property that $\lim_{s \to 0} \widetilde{\mathcal{C}}(s) = 1$ and letting $s \to 0$ in (2.40) gives

$$\pi_{\mathcal{E}} = \frac{r_2 - r_1 \lambda \mathbb{E}[B]}{r_2 + r_2 \lambda \mathbb{E}[B]}, \tag{2.41}$$

which is identical to (2.34). An alternative way to derive (2.40) is provided in Appendix A.1.

Multiplying -1 to (2.40) and taking derivatives with respect to s yields

$$-\widetilde{\mathcal{C}}'(s) = \frac{r_1^2 r_2 (r_1 + r_2) s \lambda B'(r_1 s) + r_1 r_2 (r_1 + r_2) \lambda (1 - B(r_1 s))}{\left(r_1 r_2 s - r_2 \lambda \left(1 - B(r_1 s)\right)\right)^2}, \quad s > 0. \tag{2.42}$$

Applying twice L'Hôpital's rule in (2.42) yields

$$-\lim_{s \to 0} \widetilde{\mathcal{C}}'(s) = \lim_{s \to 0} \left(\frac{r_1 r_2 (r_1 + r_2) \lambda (r_1 s B'''(r_1 s)) + B''(r_1 s))}{2[(r_1 + \lambda(1 - B(r_1 s)))^2 + (r_2 s - \lambda(1 - B(r_1 s))(r_1^2 B''(r_1 s))]} \right).$$

Simplifying the above expression, we have the expected value of the fluid level

$$\mathbb{E}[\mathcal{C}] = \frac{r_1 r_2 (r_1 + r_2) \lambda \mathbb{E}[B^2]}{2(r_2 - \lambda r_1 \mathbb{E}[B])^2}. \tag{2.43}$$

Remark 2.3.1. *Alternatively, the expression (2.40) can be achieved by using (2.21) directly. Applying the LST on both sides of (2.21) yields*

$$\mathcal{L}_f(s) = \frac{r_1 + r_2}{r_1} \mathcal{L}_{f_2}(s) \implies \widetilde{\mathcal{W}}(s) - \pi_{\mathcal{E}} = \frac{r_1 + r_2}{r_1} \left(\mathcal{L}_{f_2}(s) - \pi_{\mathcal{E}} \right), \quad s > 0, \tag{2.44}$$

where $\mathcal{L}_{f_2}(s)$ is the LST of the $f_2(x)$ in (2.19). Taking LST on both sides of (2.19) and *substituting the LST of $f_2(x)$ into the above expression yields (for $s > 0$)*

$$\widetilde{\mathcal{W}}(s) - \pi_{\mathcal{E}} = \left(\frac{r_1 + r_2}{r_1}\right) \cdot \left(\frac{\pi_{\mathcal{E}} r_2 s}{s r_2 - \lambda(1 - \widetilde{B}(r_1 s))} - \pi_{\mathcal{E}}\right) = \pi_{\mathcal{E}} \left(\frac{r_1 r_2 s + r_2 \lambda(1 - \widetilde{B}(r_1 s))}{r_1 r_2 s - r_1 \lambda(1 - \widetilde{B}(r_1 s))}\right).$$

It is important to highlight that $\mathcal{L}_{f_2}(s)$ is analogous to the Pollaczek-Khinchim formula for the virtual waiting time of an $M/G/1$ queue with two exceptions: (1) the service time of the server is the busy period of another $M/G/1$ queue with arrival rate λ/r_2 and expected service time $r_1\mathbb{E}[B]$; (2) we use the $\pi_{\mathcal{E}}$ which is the point mass for the fluid level at 0 instead of the point mass at level 0 attached to the process corresponding to $f_2(x)$.

2.3.3 Laplace-Stieltjes transform of the wet period

In this section, we explain the Laplace-Stieltjes transform of the pdf of the wet period using the probabilistic interpretation and the concepts of sub-busy periods of an $M/G/1$ queue first discussed by Kleinrock in [24, p. 210]. The concept of sub-busy periods relies on the fact that the order of arrivals being served does not affect the distribution of the busy period, as first suggested by Takacs in [39, p. 32]. Further discussions on the sub-busy periods of the $M/G/1$ queue can be found in [12, pp. 72 - 74] and [24, pp. 206 - 216].

As usual, denote by $\{\mathcal{C}(t)\}_{t\geq 0}$ the fluid level at time t, by a_i the arrivals who initiate a busy period of the driving queue in a wet period, by t_{a_i} the time at which the busy periods of the driving $M/G/1$ are initiated by a_is. It is important to highlight that the arrivals who initiate the wet periods are excluded in the definition of a_i. Finally, define

$$v_1 = a_1 \tag{2.45}$$

$$v_{n+1} = \min\{a_i \mid \mathcal{C}(t_{a_i}) < \mathcal{C}(t_{v_n}); a_i > v_n\} \tag{2.46}$$

as an arrival who initiates a sub-wet period and refer to these arrivals as tagged arrivals (see [12, Section 3.3.8, pp. 71 - 75] for the full discussion of the tagged arrivals and sub-busy periods). Define $\mathcal{N}_T$ as the number of tagged arrivals in a wet period with expected value $(r_1\lambda\mathbb{E}[B])/r_2$. Figure 2.5 (a) illustrates the concepts of the sub-wet cycle. As observed in Figure 2.5 (b), the wet period can be decomposed into

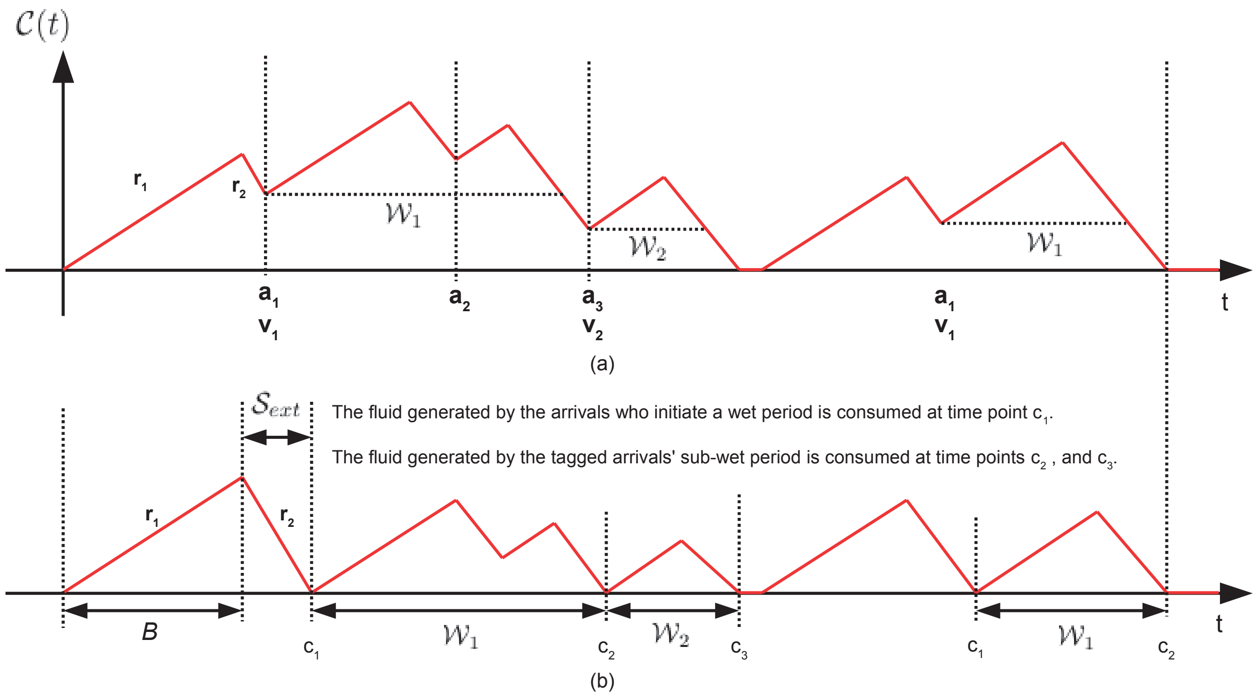

Figure 2.5: (a) Two successive wet periods of the fluid queue; (b) Life time of the sub-wet periods corresponding to (a).

$$\mathcal{W} \stackrel{dist.}{=} B + \frac{r_1}{r_2}B + \sum_{i=1}^{\mathcal{N}_T} \mathcal{W}_i, \tag{2.47}$$

where $\{\mathcal{W}_i\}_{i=1}^{\mathcal{N}_T}$ is a sequence of iid random variables distributed as $\mathcal{W}$ and independent of $\mathcal{N}_T$. Taking expectation of both sides yields

$$\mathbb{E}\left[\mathcal{W}\right] = \left(1 + \frac{r_1}{r_2}\right)\mathbb{E}[B] + \lambda\frac{r_1}{r_2}\mathbb{E}[B]\mathbb{E}[\mathcal{W}], \tag{2.48}$$

which implies that

$$\mathbb{E}[\mathcal{W}] = \frac{(r_1 + r_2)\mathbb{E}[B]}{r_2 - r_1\lambda\mathbb{E}[B]}. \tag{2.49}$$

Applying *eq. (5.2)* in [11, Boxma, Perry and Schouten, 1999] yields the following Theorem.

Theorem 2.3.2. *The Laplace-Stieltjes transform of the fluid wet period driven by an $M/G/1$ queue with arrival rate λ and expected service time $1/\mu$ satisfies the following functional equation*

$$\begin{aligned}\widetilde{\mathcal{W}}(s) &= \mathbb{E}\left[e^{-sB(1+r_1/r_2)-\lambda\left(1-\widetilde{B}(s)\right)(r_1B)/r_2}\right] \\ &= \widetilde{B}\left(s\left(1+\frac{r_1}{r_2}\right) + \lambda\left(1-\widetilde{\mathcal{W}}(s)\right)\frac{r_1}{r_2}\right), \quad s > 0,\end{aligned} \tag{2.50}$$

where r_1 and r_2 are the net input rate and leaking rate modulated by the driving process respectively, $\widetilde{\mathcal{W}}(s)$ and $\widetilde{B}(s)$ are the Laplace-Stieltjes transform of the wet period and the busy period of the $M/G/1$ queue respectively.

The details of the proof are skipped by the authors in [11]. Hence, we provide a proof for Theorem 2.3.2 in the Appendix A.2. Here, we use the probabilistic interpretation of the LST to interpret the LST of the pdf of the wet period.

Before going to the probabilistic interpretation, we first introduce an extended silence period of the fluid queue. Consider at time 0, an arrival ends the empty period of the fluid queue and initiates a busy period B in the driving queue that contributes r_1B units of the fluid at the end of the busy period. After the end of the busy period, the fluid queue takes $(r_1B)/r_2$ units of time to release the fluid that is generated by the busy period. We refer to the $(r_1B)/r_2$ units of time as an extended silence period. It is important to highlight that the number of tagged arrivals (i.e. the number of sub-wet periods) in a wet period is identically distributed as the number of arrivals in the extended silence period. Figure 2.5 (b) illustrates the concept of the extended silence period. The extended silence period is indicated as $\mathcal{S}_{ext}$ in Figure 2.5 (b).

Denote by $\widetilde{\mathcal{W}}(s)$ the LST of the pdf of the wet period. Each tagged arrival entering the driving queue in the extended silence period has a sub-wet period associated with it. With probability $\widetilde{\mathcal{W}}(s)$, the sub-wet period initiated by a tagged arrival ends before the catastrophe occurs. Similarly, with probability $1-\widetilde{\mathcal{W}}(s)$, the sub-wet period initiated by a tagged arrival ends after the catastrophe occurs. We refer to a good arrival if the associated sub-wet period ends before the catastrophe occurs, and refer to a bad arrival if the associated sub-wet period ends after the catastrophe occurs. Multiplying λ to the probabilities $\widetilde{\mathcal{W}}(s)$ and $1-\widetilde{\mathcal{W}}(s)$ gives us the rate $\lambda\widetilde{\mathcal{W}}(s)$ at which the good arrivals arrive, and the rate $\lambda(1-\widetilde{\mathcal{W}}(s))$ at which the bad arrivals arrive.

Following the probabilistic interpretation of the LST of the pdf of the busy period in [35], the LST of the pdf of the wet period can be defined as

$$\begin{aligned}
\widetilde{\mathcal{W}}(s) &= \mathbb{P}[\text{wet period ends before the catastrophe occurs}] \\
&= \mathbb{P}[\text{first activity period ends before the catastrophe occurs,} \\
&\qquad \text{extended silence period ends before catastrophe occurs} \\
&\qquad \text{and ends before bad arrival arrives}].
\end{aligned}$$

Define the random variable Y as a catastrophe random variable which is exponentially distributed with rate s for the first activity and extended silence period, and the random variable Y_A as the time that a bad arrival arrives which is exponential distributed with rate $\lambda(1-\widetilde{\mathcal{W}}(s))$. Applying a linear transform technique, we obtain

$$\begin{aligned}\widetilde{\mathcal{W}}(s) &= \mathbb{P}[\min(Y,(r_2/r_1)Y,(r_2/r_1)Y_A) > B] \\ &= \widetilde{B}\left(s\left(1+\frac{r_1}{r_2}\right)+\frac{r_1}{r_2}\lambda(1-\widetilde{\mathcal{W}}(s))\right), \quad s>0.\end{aligned}$$

which is equivalent to Theorem 2.3.2, where $\widetilde{B}(s)$ is the LST of the pdf of the busy period of the driving queue.

2.4 Number of tagged arrivals, arrivals served, and peaks in a wet period

2.4.1 Number of tagged arrivals in a fluid wet cycle

In this section, we derive the probability generating function (hereafter pgf) of the number of tagged arrivals $\mathcal{N}_T$ in a wet cycle. The tagged arrivals provide an important performance measure for a fluid queue, since the duration of the wet period depends on the number of tagged arrivals in the extended silence period. Let B be the busy period of the driving $M/G/1$ queue. Then the extended silence period is $\mathcal{S}_{ext} = (r_1 B)/r_2$ (see Figure 2.5 for the details). Thus, the conditional random variable $(\mathcal{N}_T|B=x)$ is Poisson distributed with parameter $(r_1 x\lambda)/r_2$. Consequently, the probability of the number of tagged arrivals in a wet period is

$$\mathbb{P}[\mathcal{N}_T=n] = \int_0^\infty \mathbb{P}[\mathcal{N}_T=n|B=x]d\mathbb{P}[B<x], \quad n=0,\ 1,\dots \tag{2.51}$$

where

$$\mathbb{P}[\mathcal{N}_T = n | B = x] = \frac{\exp(-r_1 x \lambda / r_2)(r_1 x \lambda / r_2)^n}{n!}. \tag{2.52}$$

Multiplying both sides of (2.51) by z^n, summing over n, and using Taylor expansion of the exponential function, we have the pgf of the number of tagged arrivals in a wet period is

$$m_{\mathcal{N}_T}(z) = \widetilde{B}\left(\frac{r_1}{r_2}\lambda(1-z)\right), \quad 0 < z < 1, \ x > 0. \tag{2.53}$$

The same procedures can be used to find the number of tagged arrivals for an $M/G/1$ queue (see [18, p. 226, and p. 238] for the details).

2.4.2 Number of arrivals served in a wet cycle

Denote by $\mathcal{N}_B$ the number of arrivals served in the driving $M/G/1$ queue during of a wet cycle. Since each tagged arrival initiates a sub-wet period of the fluid queue, the number of arrivals served in the wet period is the number of of arrivals served in the first activity period plus the number of arrivals served in each sub-wet period,

$$\mathcal{N}_B = N + \mathcal{N}_B^1 + \cdots + \mathcal{N}_B^{\mathcal{N}_T}, \tag{2.54}$$

where N is the number of arrivals served in the first activity period (or equivalently, the number of arrivals served in the busy period that initiates the wet period), and $\mathcal{N}_B^i, i = 1, \ 2, \ldots, \mathcal{N}_T$, are the number of arrivals served in each sub-wet period. The $N_B^i s$ are independent and identically distributed as $\mathcal{N}_B$. We remark here that $\mathcal{N}_T$ can take an integer value from 0 to ∞. When $\mathcal{N}_T = 0$, then $\mathcal{N}_B = \mathcal{N}_T$. Taking expectation in both sides of (2.54) and applying Little's law (i.e., $\mathbb{E}[\mathcal{N}_T] = (\lambda/r_2) \cdot (r_1 \mathbb{E}[B])$) and Wald's theorem ([45]), we obtain

$$\mathbb{E}[\mathcal{N}_B] = \mathbb{E}[N] + \mathbb{E}\left[\sum_{i=0}^{\mathcal{N}_T} \mathcal{N}_B^i\right] = \frac{1}{\pi_0} + \lambda\frac{r_1}{r_2}\mathbb{E}[B]\mathbb{E}[\mathcal{N}_B],$$

which implies that

$$\mathbb{E}[\mathcal{N}_B] = \frac{1}{1-\lambda\frac{r_1}{r_2}\mathbb{E}[B]} \cdot \frac{1}{\pi_0} = \mathbb{E}[\mathcal{N}_p]\mathbb{E}[N], \tag{2.55}$$

where $\pi_0 := 1 - \lambda\mathbb{E}[S]$ is the point mass of the driving $M/G/1$ is empty, where S is the service time of the driving $M/G/1$ queue. Next define the pgf of the number of arrivals served in a wet cycle as

$$m_{\mathcal{N}_B}(z) = \mathbb{E}\left[z^{\mathcal{N}_B}\right], \quad 0 < z < 1. \tag{2.56}$$

Conditioning the above expression on $\mathcal{N}_T = k$ yields

$$\mathbb{E}\left[z^{\mathcal{N}_B} | \mathcal{N}_T = k\right] = \mathbb{E}\left[z^N\right] \prod_{i=1}^{k} E\left[z^{\mathcal{N}_B^i}\right] = \mathbb{E}\left[z^N\right] \left(m_{\mathcal{N}_B}(z)\right)^k. \tag{2.57}$$

Substituting equation (2.54) into equation (2.56) and using equation (2.57) yields

$$m_{\mathcal{N}_B}(z) = \mathbb{E}\left[z^{N+\mathcal{N}_B^1+\cdots+\mathcal{N}_B^{\mathcal{N}_T}}\right] = m_N(z) m_{\mathcal{N}_T}(m_{\mathcal{N}_B}(z)), \quad 0 < z < 1,$$

where $m_N(z)$ and $m_{\mathcal{N}_T}(z)$ are the pgfs of the number of arrivals served in an $M/G/1$ queue busy period and the number of tagged arrivals in a wet cycle respectively. Further simplification can be achieved by using the following equality in [31, p. 307, eq. (7.47)]

$$m_{\mathcal{N}_T}(z) = \widetilde{G}_S\left(\lambda(1-z)\right), \quad 0 < z < 1,$$

where $\widetilde{G}_S$ is the LST of the pdf of the service time of an $M/G/1$ queue with arrival rate λ and service rate μ. Finally, we have

$$m_{\mathcal{N}_B}(z) = m_N(z)\widetilde{G}_S\left(\lambda_f(1 - m_{\mathcal{N}_B}(z))\right), \quad 0 < z < 1. \tag{2.58}$$

Here the busy period of a regular $M/G/1$ queue with arrival rate $\lambda_f := \lambda/r_2$ and service rate $\mu_f := 1/(r_1\mathbb{E}[B])$ is served as a service time for the $M/G/1$ queue in (2.58). Taking derivatives with respect with z in (2.58) yields

$$m'_{\mathcal{N}_B}(z) = -\frac{\lambda}{r_2}m_N(z)\widetilde{G}'_S\left(\frac{\lambda}{r_2}(1 - m_B(z))\right)m'_{\mathcal{N}_B}(z) + \widetilde{G}_S\left(\frac{\lambda}{r_2}(1 - m_B(z))\right)m'_N(z).$$

Substituting $z = 1$ into the above equation yields

$$m'_{\mathcal{N}_B}(1) = \frac{\lambda}{r_2}m_N(1)\widetilde{G}'_S(0)m'_{\mathcal{N}_B}(1) + \widetilde{G}_S(0)m'_N(1).$$

Simplifying the above expression and collecting the terms $\mathbb{E}[\mathcal{N}_B]$ yields the same expression as shown in (2.55). The immediately above expression is a check on the general formula in (2.58).

2.4.3 Number of peaks in a wet cycle

Denote by $\mathcal{N}_P$ the number of busy periods in a wet period (and thus the number of peaks in a wet cycle). Using a similar argument as in Section 2.4.2, we express the number of driving queue $M/G/1$ busy periods in a wet period in terms of the number of busy periods in each sub-wet period and the busy period corresponding to the activity period that initiates the wet period. Hence, we have

$$\mathcal{N}_P = 1 + \mathcal{N}_P^1 + \cdots + \mathcal{N}_P^{\mathcal{N}_T}, \tag{2.59}$$

where $\mathcal{N}_T$ is the number of tagged arrivals in a fluid wet period, $\mathcal{N}_P^i, i = 1, 2, \ldots, \mathcal{N}_T$ is the number of driving $M/G/1$ queue busy periods within a sub-wet period that are independent and identically distributed as $\mathcal{N}_P$. Define the pgf of the number of peaks in a wet cycle as

$$m_{\mathcal{N}_P}(z) = \mathbb{E}\left[z^{\mathcal{N}_P}\right]. \tag{2.60}$$

Substituting equation (2.59) into equation (2.60) yields

$$m_{\mathcal{N}_P}(z) = \mathbb{E}\left[z^{1+\mathcal{N}_P^1+\cdots+\mathcal{N}_P^{\mathcal{N}_T}}\right] = z\widetilde{G}_S(\lambda_f(1 - m_{\mathcal{N}_P}(z))), \quad 0 < z < 1,$$

where $m_{\mathcal{N}_T}(z)$ is the pgf of the number of arrivals served in an $M/G/1$ queue busy period, and $\widetilde{G}_S(z)$ is the LST of the pdf of the busy period of the $M/G/1$ queue with arrival rate $\lambda_f := \lambda/r_2$ and service rate $\mu_f := 1/(r_1\mathbb{E}[B])$. Taking the derivative of $m_{\mathcal{N}_P}(z)$ with respect to z and evaluating at $z \to 1$ gives

$$\begin{aligned} m'_{\mathcal{N}_P}(z)\mid_{z=1} &= \widetilde{G}_S\left(\lambda_f(1 - m_{\mathcal{N}_P}(z))\right)\mid_{z=1} - \lambda_f z\widetilde{G}_S\left(\lambda_f(1 - m_{\mathcal{N}_P}(z))\right) m'_{\mathcal{N}_P}(z)\mid_{z=1} \\ &= \widetilde{G}_S(0) - \lambda_f \widetilde{G}'_S(0) m'_{\mathcal{N}_P}(1), \end{aligned}$$

which leads to

$$\mathbb{E}[N_{\mathcal{N}_P}] = 1 + \lambda_f E[S]\mathbb{E}[\mathcal{N}_P] = \frac{1}{1 - \frac{r_1}{r_2}\lambda\mathbb{E}[B]}. \tag{2.61}$$

The expression (2.61) confirms the expression (2.17) derived from the analysis of the sample path of the fluid level in Section 2.1.3.

2.5 Numerical illustration

2.5.1 Simulation

In this section, we present a numerical example which illustrates the expected value of the fluid level and the expected value of the wet period using **Theorem 2.3.2**, and the LSTs in (2.40) and (2.50) respectively. For each iteration and each set of parameters, we simulate $10,000$ arrivals and record the duration of the wet periods, wet cycles, and the average fluid level in each wet cycle. The simulations are conducted using Maple 18. Note that, the stability condition of the fluid queue is

$$r_1\mathbb{E}[B] < r_2\mathbb{E}[I] \iff \alpha := \left(\frac{r_1}{r_2}\right)\left(\frac{\lambda}{\mu-\lambda}\right) < 1. \tag{2.62}$$

Thus to satisfy the stability condition, we choose the combination of r_1, r_2, λ, and μ such that $\alpha < 1$. The results and the parameters for each simulation are reported in Table 2.1. It is important to note that as $\alpha \to 1$, the results achieved using (2.40) and (2.50) might not be close to the simulated results because many samples are required under this case. We choose parameter values which cover a large spectrum of values of α by carefully selecting values r_1, r_2, λ, and μ which satisfy the condition (2.62). As observed in Table 2.1, as α equals 0.90 and 0.89, the simulated and the theoretical expected values are off by 6% to 3% (with respect to the simulated results) for the fluid level respectively, whereas simulated wet periods are still close to the theoretical expected wet period (off by 1.41% and 0.03% with respect to the simulated results).

Table 2.1: Simulation results of fluid queue expected values

Parameters					Simulated results			Theoretical results		
λ	μ	r_1	r_2	α	$\mathbb{E}[\mathcal{C}]$	$\mathbb{E}[\mathcal{B}]$	$\pi_{\mathcal{E}}$	$\mathbb{E}[\mathcal{C}]$	$\mathbb{E}[\mathcal{B}]$	$\pi_{\mathcal{E}}$
2.5	7.0	1.0	2.0	0.28	0.255	0.461	0.464	0.256	0.461	0.464
3.0	7.0	1.0	2.0	0.38	0.454	0.601	0.356	0.449	0.600	0.357
3.5	7.0	1.0	2.0	0.50	0.835	0.845	0.252	0.857	0.857	0.250
4.0	7.0	1.0	2.0	0.67	1.919	1.484	0.144	1.999	1.500	0.142
4.5	7.0	1.0	2.0	0.90	10.09	5.915	0.036	10.79	6.000	0.035
2.5	6.0	1.0	2.0	0.36	0.477	0.667	0.374	0.476	0.666	0.375
2.5	5.5	1.0	2.0	0.42	0.708	0.852	0.319	0.714	0.857	0.318
2.5	5.0	1.0	2.0	0.50	1.170	1.184	0.252	1.199	1.200	0.250
2.5	4.5	1.0	2.0	0.63	2.570	2.009	0.165	2.449	2.000	0.166
2.5	4.0	1.0	2.0	0.83	9.599	5.998	0.062	9.999	6.000	0.062
2.5	6.0	0.5	2.0	0.18	0.154	0.435	0.479	0.155	0.434	0.479
2.5	6.0	1.0	2.0	0.36	0.480	0.667	0.374	0.476	0.666	0.375
2.5	6.0	1.5	2.0	0.54	1.130	1.064	0.272	1.153	1.076	0.270
2.5	6.0	2.0	2.0	0.71	2.848	1.986	0.168	2.857	2.000	0.166
2.5	6.0	2.5	2.0	0.89	10.31	5.988	0.062	10.71	6.000	0.062
2.5	6.0	1.0	4.0	0.18	0.312	0.435	0.479	0.310	0.434	0.479
2.5	6.0	1.0	3.5	0.20	0.328	0.459	0.465	0.329	0.461	0.464
2.5	6.0	1.0	3.0	0.24	0.353	0.497	0.357	0.357	0.500	0.444
2.5	6.0	1.0	2.5	0.29	0.403	0.560	0.415	0.399	0.560	0.416
2.5	6.0	1.0	2.0	0.36	0.471	0.667	0.375	0.476	0.666	0.375

2.6 Relating probability distributions of the driving and fluid queues

In this section, we establish a relationship between the pdf of the virtual waiting time of the driving queue system and the pdf of the fluid level of the fluid queue. To do so, we first observe the following property. By letting $r_1 = r_2 = 1$ in (2.19), we have

$$f_2(x) = \lambda \pi_{\mathcal{E}} \overline{B}(x) + \lambda \int_{y=0^+}^{y=x} \overline{B}\left(x-y\right) f_2(y) dy, \tag{2.63}$$

which is the integral equation for the pdf of the virtual waiting time of an $M/G/1$ queue with arrival rate λ and a specific service time which is distributed as a busy period of an $M/G/1$ queue with arrival rate λ and expected service time $1/\mu$. It is important to highlight that the term $\pi_{\mathcal{E}}$ in (2.63) corresponding to the point mass of fluid level at 0 rather than the point mass of virtual waiting time of the $M/G/1$ queue at 0.

Denote by $G(x)$ the distribution of the general service time of an $M/G/1$ queue. Replacing $\overline{B}(x)$ by $\overline{G}(x)$ in (2.63), then (2.63) becomes the equation for the pdf of the virtual waiting time of an $M/G/1$ queue[3] with an exception that the term $\pi_{\mathcal{E}}$ corresponding to the point mass of fluid level at 0. By (2.22) with the assumption that $r_1 = r_2 = 1$, we have $f_2(x) = f(x)/2$, where $f(x)$ is the steady-state distribution of the fluid level. These properties together give us

$$\mathcal{L}_{f_2}(s) = \frac{1}{2}\left(\widetilde{\mathcal{C}}(s) - \pi_{\mathcal{E}}\right) + \pi_{\mathcal{E}} = \frac{s\pi_{\mathcal{E}}}{s - \lambda\left(1 - \widetilde{G}(s)\right)}, \quad s > 0 \tag{2.64}$$

which is the LST of the pdf of the virtual waiting time of an $M/G/1$ queue. Next, denote by π_0 the point mass of virtual waiting time of an $M/G/1$ queue at 0. Replacing $\pi_{\mathcal{E}}$ by π_0 and taking $s \to 0$ in (2.64) yield

$$\pi_0 = 1 - \lambda/\mu. \tag{2.65}$$

Remark 2.6.1. *The LST of the virtual waiting time of an $M/G/1$ queue $f_2(x)$ is evaluated using the LST of the pdf of the fluid level $\widetilde{\mathcal{C}}(s)$ which involves $\pi_{\mathcal{E}}$ as indicated in (2.64). After we find the LST of $f_2(x)$, we substitute π_0 for $\pi_{\mathcal{E}}$. Since, we are matching the LC equation of the pdf of the virtual waiting time of an $M/G/1$ queue to the LC equations of $f_2(x)$ in (2.64).*

Thus, studying the distribution of the fluid level leads us to the distribution of the fluid level as well as the distribution of the virtual waiting time of the driving $M/G/1$ queue.

[3] Here $\overline{B}(x)$ is just a dummy variable attached with a specific distribution. Changing the notation of B does not destroy the structure of the LC equation.

In the next two subsections, we study the distribution of the fluid level which is driven by two $M/G/1$ variants and highlight the distribution of the waiting time associated with the distribution of the fluid level in the fluid queue.

2.6.1 Fluid queue driven by modified $M/G/1$ queue

In this example, we use a fluid queue model to derive the pdf of the virtual waiting time of an $M/G/1$ queue with multiple inputs discussed in [12, Section 3.6]. The characteristics of the queue are similar to the ordinary $M/G/1$ queue except arrivals arrive to the system in N independent Poisson processes with rate λ_i, and the service times associated to the N independent Poisson arrivals follows $G_i, i = 1, \ldots, N$, distribution. The LC equation corresponding to the pdf of the waiting time of the model has been shown in [12, eq. (3.125), p.103] to be

$$f(x) = \lambda \sum_{i=1}^{N} \frac{\lambda_i}{\lambda} \left(\pi_0 \overline{G}_i(x) + \int_{y=0}^{x} \overline{G}_i(x-y) f(y) dy \right), \tag{2.66}$$

where $\lambda = \lambda_1 + \cdots + \lambda_N$, and $\overline{G}_i(\bullet)$ is the ccdf of the service time for the arrivals coming from the i^{th} source of the Poisson process, and π_0 is the probability point mass of waiting time at level 0.

To construct a fluid queue that shares similar characteristics of the $M/G/1$ queue with multiple inputs, we first note that the activity period $\mathcal{A}_i$ of the fluid queue multiplied by the *net* input rate r_1 of the fluid queue gives the service time of the $M/G/1$ queue with multiple inputs, when we use the pdf of the fluid level of the fluid queue to derive the pdf of the wait of the multiple inputs model. Next, as observed in Figure 2.6 below, the duration of the activity period $\mathcal{A}_i$, $i = 1, \ldots, N$ of the fluid queue is equal to the duration of the busy period $B_i^{(j)}$, $j = 1, \ldots, N$ of the driving queue. Hence, to capture the characteristic of the $M/G/1$ queue with multiple inputs such that the service time of the arrivals depends on the sources of the Poisson process, we have to restrict the service times of the arrivals in the same activity period to follow a common distribution. In addition, in the fluid queue, we assume that all arrivals follow a Poisson process with rate λ instead of multiple Poisson inputs with different rates. Keeping this in mind, Figure 2.6 illustrates a sample path of the fluid queue and the virtual waiting time of the driving process. As observed in Figure 2.6, arrivals arrive to the system at time points t_1 and t_4, the driving queue manager assigns these arrivals a service time which follows a distribution $G^{(1)}$ with rate $1/\mu_1$ with probability λ_1/λ, such that $\lambda = \lambda_1 + \cdots + \lambda_N$. These two arrivals initiate busy periods $B_1^{(1)}$ and $B_1^{(4)}$ in the driving queue and activity periods $\mathcal{A}_1$ and $\mathcal{A}_4$ in the fluid queue. In addition, all other arrivals in $B_1^{(1)}$ and $B_4^{(1)}$ receive a service time that follows a common distribution that the first arrivals in $B_1^{(1)}$ and $B_1^{(4)}$ encountered.

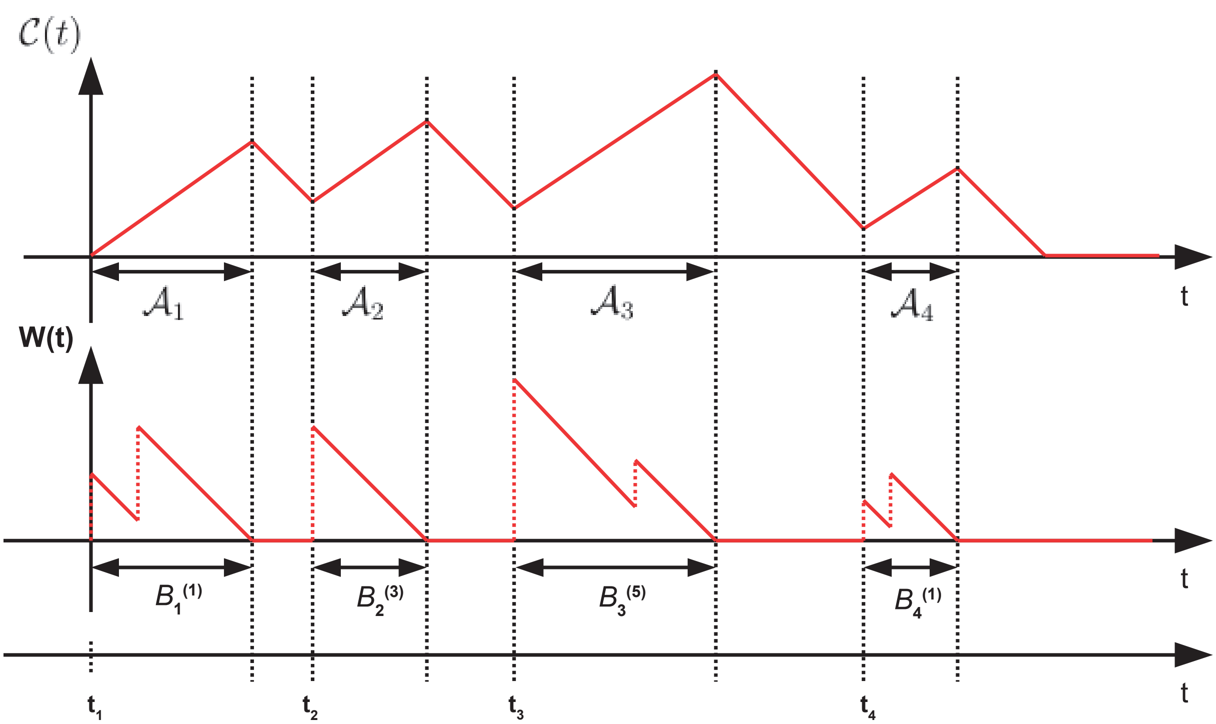

Figure 2.6: Sample path of the fluid level and waiting time.

Denote by $\overline{B}^{(i)}(x)$ the ccdf of the busy period of the driving queue that the manager assigns to the first arrival in the busy period. Using the expression (2.23), for $x > 0$, we have

$$f(x) = \left(\frac{1}{r_1} + \frac{1}{r_2}\right) \cdot \pi_{\mathcal{E}} \cdot \sum_{i=1}^{n} \lambda_i \overline{B}^{(i)}\left(\frac{x}{r_1}\right) + \frac{1}{r_2} \sum_{i=1}^{n} \lambda_i \int_{y=0^+}^{\infty} \overline{B}^{(i)}\left(\frac{x-y}{r_1}\right) f(y) dy. \quad (2.67)$$

Applying the LST operator on both sides of (2.67) yields

$$\widetilde{\mathcal{C}}(s) = \pi_{\mathcal{E}} \frac{r_1 r_2 s + r_2 \sum_{i=1}^{n} \lambda_i (1 - \widetilde{B}^{(i)}(r_1 s))}{r_1 r_2 s - r_1 \sum_{i=1}^{n} \lambda_i (1 - \widetilde{B}^{(i)}(r_1 s))}, \quad s > 0, \quad (2.68)$$

where

$$\pi_{\mathcal{E}} = \frac{r_2 - r_1 \sum_{i=1}^{n} \lambda_i \mathbb{E}[B^{(i)}]}{r_2 + r_2 \sum_{i=1}^{n} \lambda_i \mathbb{E}[B^{(i)}]}. \quad (2.69)$$

Using the similar arguments as in (2.64) and (2.65), i.e. $r_1 = r_2 = 1$, the LST of the pdf of the virtual waiting time of the $M/G/1$ queue with multiple inputs (denoted as $\mathcal{L}_g(s)$), associated with the fluid queue is

$$\mathcal{L}_g(s) = \frac{1}{2}\left(\mathcal{L}_f(s) - \pi_{\mathcal{E}}\right) + \pi_{\mathcal{E}} = \frac{s\pi_{\mathcal{E}}}{s - \sum_{i=1}^{n} \lambda_i (1 - \widetilde{G}^{(i)}(s))}, \quad s > 0. \quad (2.70)$$

Denote by π_0 the point mass of the virtual waiting time of the $M/G/1$ queue with multiple Poisson inputs. Substituting $\pi_{\mathcal{E}}$ by π_0 in (2.70) and letting $s \to 0$ yield

$$\pi_0 = 1 - \sum_{i=1}^{n} \lambda_i / \mu_i. \quad (2.71)$$

The expression (2.70) is the LST of the pdf of the virtual waiting time of an $M/G/1$ with multiple Poisson inputs discussed in [12, Section 3.6, pp. 102 - 106], and π_0 (replacing for $\pi_{\mathcal{E}}$) is the probability of the virtual waiting time being at level 0 for the multiple Poisson inputs queue. For the details of the queueing system, please refer to [12, Section 3.6, pp. 102 - 106].

2.6.2 Fluid queue driven by an $M/M/1/1$ queue

In this section, we first consider a fluid queue driven by an $M/G/1$ queue with arrival rate λ and expected service time $1/\mu$. In addition, we assume that zero-wait arrivals (an arrival's waiting time is 0) at time t, join the driving $M/G/1$ queue only if the fluid level of the fluid queue at time t is below K; otherwise, these zero-wait arrivals will be blocked by the system manager from entering the driving queue. We refer to this discipline as a *balking rule*. In other words, a busy period of the driving $M/G/1$ queue can be initiated only if the fluid level is below level K. However, the fluid level can go above level K if the busy period of the driving queue is initiated below level K.

Figures 2.7 (a) and (b) illustrate the sample path of the fluid level and the virtual waiting time V_b of the driving $M/G/1$ queue. Here, zero-wait arrivals at time point t_1 and t_2 are blocked from entering the driving queue, since the fluid levels $\mathcal{C}$ at time point t_1 and t_2 are above level K. It is important to observe in Figures 2.7 (a) and (b) that the busy periods in the driving queue are distributed as a regular busy period of an $M/G/1$ queue. Blocking the zero-wait arrivals do not affect the characteristics of the busy period of the driving queue. Hence, using the previous result (2.23), we have the following LC equations:

$$g_1(x) = \lambda\pi_{\mathcal{E}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\overline{B}\left(\frac{x}{r_1}\right)+\frac{\lambda}{r_2}\int_{y=0^+}^{y=x}\overline{B}\left(\frac{x-y}{r_1}\right)g_1(y)dy,\ 0<x\le K, \tag{2.72}$$

$$g_2(x) = \lambda\pi_{\mathcal{E}}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\overline{B}\left(\frac{x}{r_1}\right)+\frac{\lambda}{r_2}\int_{y=0^+}^{y=K}\overline{B}\left(\frac{x-y}{r_1}\right)g_2(y)dy,\ K<x<\infty, \tag{2.73}$$

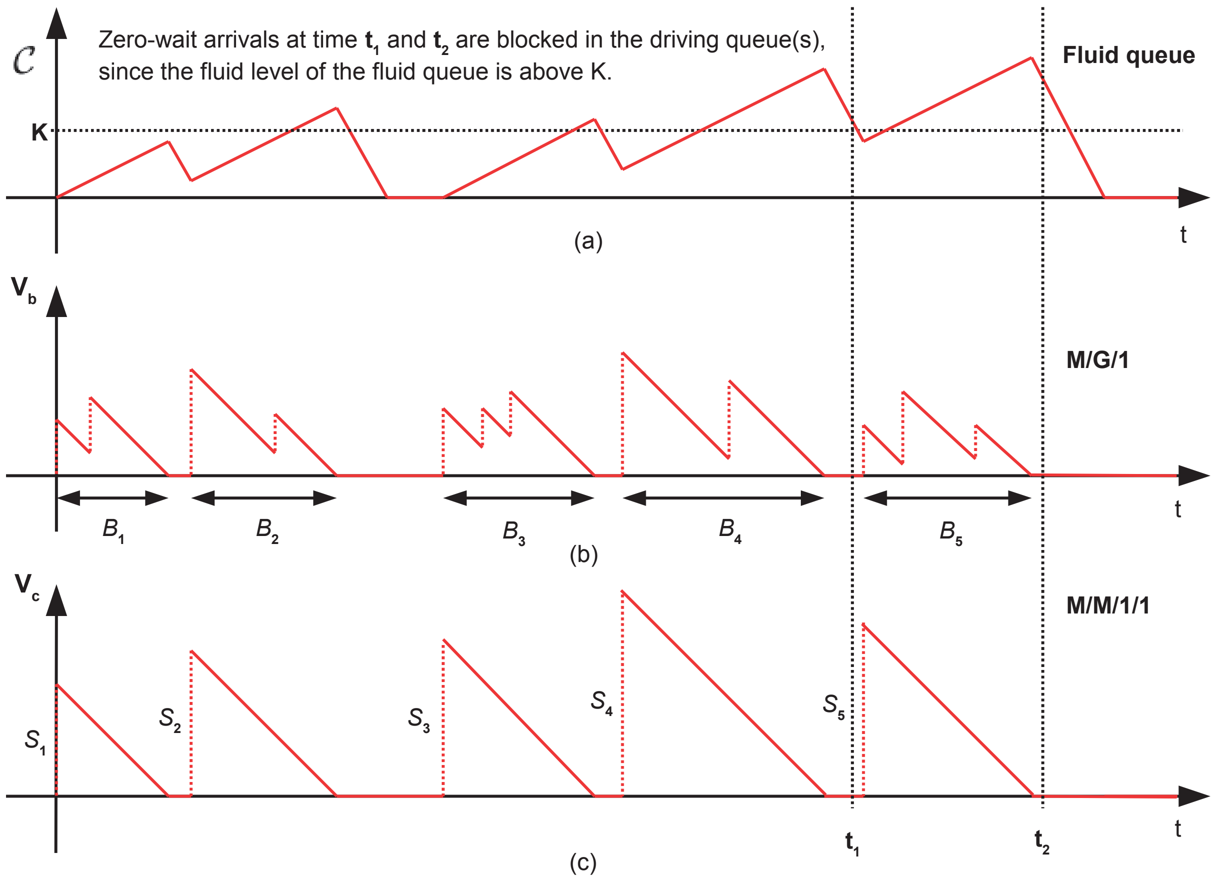

Figure 2.7: Sample paths of the fluid level and corresponding virtual waiting time of the driving queue: Diagrams (a), (b) and (c) are discussed below.

with the following initial conditions:

$$g_1(0^+) = \lambda \pi_{\mathcal{E}} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \quad \text{and} \quad g_1(K^-) = g_2(K^+), \tag{2.74}$$

where the continuity at the level K is due to the continuity of the sampe path at every up- and downcrossing instant of level K, and there are no sample path tangents at level K (see [12, Proposition 3.7, p. 113, for $M/D/1$]). Solving the equations (2.72) and (2.73) is difficult since we do not know the form of $\overline{B}(x)$ which is the ccdf of the busy period of the driving $M/G/1$ queue.

Driving queue: $M/M/1/1$ queue

Instead of focusing on the driving process as an $M/G/1$ queue with general 'G', we now assume that the driving process is an $M/M/1/1/$ queue. The details of the $M/M/1/1$ queue can be found in [12, Section 3.4.10], [24, Section 3.6], and [41, Chapter 5]. In addition, we apply the same *balking rule* to this fluid queue, as above. The sample paths of the fluid level and virtual waiting time of the driving $M/G/1$ queue are illustrated in Figure 2.7 (a) and (c). As observed, similar to the $M/G/1$ queue example, zero-wait arrivals at time points t_1 and t_2 are blocked from entering the driving queue by the system manager. Since we assume the service time S of the $M/M/1/1$ queue is $Exp(\mu)$, we have $\mathbb{P}[S > t] = \exp(-\mu t)$. Substituting this into (2.72) and (2.73), we have (for $x \leq K$)

$$g_1(x) = \lambda \pi_{\mathcal{E}} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \exp\left(-\frac{\mu}{r_1} x \right) + \frac{\lambda}{r_2} \int_{y=0^+}^{y=x} g_1(y) \exp\left(-\frac{\mu}{r_1}(x-y) \right) dy. \tag{2.75}$$

Similarly, for $x > K$, we have,

$$g_2(x) = \lambda \pi_{\mathcal{E}} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \exp\left(-\frac{\mu}{r_1} x \right) + \frac{\lambda}{r_2} \int_{y=0^+}^{y=K} g_2(y) \exp\left(-\frac{\mu}{r_1}(x-y) \right) dy. \tag{2.76}$$

The interpretation of the (2.75) and (2.76) are the same as for (2.23), except that the upper limit of the integrals are restricted by the level of $y \leq K$. To solve for $f_1(x)$ and $f_2(x)$ in (2.75) and (2.76), we take derivative with respect to x in (2.75) and (2.76). This gives us

$$g_1'(x) = -\left(\frac{\mu}{r_1} - \frac{\lambda}{r_2} \right) g_1(x) \quad \text{and} \quad g_2'(x) = -\frac{\mu}{r_1} g_2(x) \tag{2.77}$$

with a solution that

$$g_1(x) = C_1 \exp\left(-\left(\frac{\mu}{r_1} - \frac{\lambda}{r_2}\right)x\right), \quad x \leq K, \tag{2.78}$$

$$g_2(x) = C_2 \exp\left(-\frac{\mu}{r_1}x\right), \quad x > K. \tag{2.79}$$

Using the initial condition (2.74) with (2.79), we have

$$C_1 = \lambda\pi_{\mathcal{E}}\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \quad \text{and} \quad C_2 = \lambda\pi_{\mathcal{E}}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)\exp\left(\frac{\lambda}{r_2}K\right). \tag{2.80}$$

Finally, using the normalizing condition,

$$\pi_{\mathcal{E}} + \int_{0^+}^{\infty} g_1(x) + g_2(x)dx = 1, \tag{2.81}$$

we have

$$\pi_{\mathcal{E}} = \frac{r_2(r_2 - r_1\rho)}{(r_1 + r_2)\rho\left(r_2 - \rho r_1 \exp\left(-\left(\frac{\mu}{r_1} - \frac{\lambda}{r_2}\right)K\right)\right)}, \tag{2.82}$$

where $\rho := \lambda/\mu$. To check that $\pi_{\mathcal{E}}$ satisfies a necessary condition, we evaluate the units on both sides of (2.82). We obtain

$$[\pi_{\mathcal{E}}] = \left[\frac{(gram/time)^2}{(gram/time)^2}\right] = \text{dimensionless}, \tag{2.83}$$

which is a correct dimension for a probability value. This suggests that the expression (2.82) is reasonable.

To recap our findings in this example, we find the pdf of the fluid level modulated by the

$M/M/1/1$ queue with arrival rate λ and service rate μ,

$$f(x) = \begin{cases} \lambda \pi_{\mathcal{E}} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \exp\left(- \left(\frac{\mu}{r_1} - \frac{\lambda}{r_2} \right) x \right), & x \leq K \\ \\ \lambda \pi_{\mathcal{E}} \left(\frac{1}{r_1} + \frac{1}{r_2} \right) \exp\left(\frac{\lambda}{r_2} K \right) \exp\left(- \frac{\mu}{r_1} x \right), & x > K \end{cases} \tag{2.84}$$

where $\pi_{\mathcal{E}}$ is defined in (2.82).

In addition, letting $r_1 = 1$, $r_2 = 1$, multiplying (2.84) by 1/2 and evaluating[4] $\pi_{\mathcal{E}}$ in (2.82), we get the steady-state distribution of waiting time of an $M/M/1$ queue with workload dependent balking [29, *Theorem 4*, p. 256]. In addition, letting $K \to \infty$, the expression (2.84) becomes the distribution of the virtual waiting time of an $M/M/1$ queue [12, eq. (3.87) - (3.88), p. 86]. Finally, these two examples hint that we can find the distribution of the waiting time of an $M/G/1$-like queue by matching the $M/G/1$-like queue to the fluid queue. Finally, it is interesting to point out that the driving queue in this example is an $M/M/1/1$ queue; however, we express the pdf of the waiting time of the $M/G/1$ queue with balking rather than the pdf of the waiting time of the $M/M/1/1$ queue by mapping the fluid queue to the $M/G/1$ queue with balking.

[4] The authors have a typo on c (which is $2 \cdot \pi_{\mathcal{E}}$ in our example) on *Theorem 6* in [28, p. 27]. When $\rho \neq 1$, $c = (1-\rho)/(\rho - \rho^2 e^{-(\mu-\lambda)b})$ instead of $c = (1-\rho)/(1-\rho^2 e^{-(\mu-\lambda)b})$, where $b = K$ in our case.

Chapter 3

Fluid queue with level-dependent rates

We consider two fluid queues with infinite capacity driven by an $M/G/1$ queue with *net* input and the *continuous* leaking rates depending on the fluid level of the fluid queue and the type of arrivals in the driving queue. The first model considered in this chapter is a fluid queue such that the leaking rate depends on the fluid level $x > 0$. The second model considered in this chapter is a fluid queue such that both the *net* input and *continuous* leaking rates depend on *type of arrivals*. The structure of this chapter is as follows: Section 3.1 corresponds to the first model. There, we first define a fluid queue such that the leaking rate depends on the fluid level, and then we study the characteristics of such a model in Sections 3.1.2 - 3.1.4. In Sections 3.1.5 and 3.1.6, we provide two examples for this model. Section 3.2 corresponds to the second model. The structure of this section is similar to Section 3.1, except we provide a simulation results in Section 3.2.5 instead of providing examples as in Sections 3.1.5 and 3.1.6.

3.1 Fluid queue with leaking rate depending on fluid level

3.1.1 Introduction

In this section, we consider a fluid queue with a constant *net* input rate k_1 and leaking rate k_2x that depends on the fluid level x (> 0). For example, the total input rate may be $k_1 + k_2x$, in order to compensate for the *continuing* leakage rate k_2x. In terms of application, such a fluid queue can be used to model battery life of an electronic device. For instance, the battery life of a device might be modulated by a program within the device that depends on the remaining life time of the battery before the next charge. To be more specific, a cell phone device might contain a program to detect the remaining life time of the battery. When the lifetime of the battery is high, the cell phone is fully functioning; if the lifetime of the battery is low, certain functions of the cell phone are restricted to access to reduce the consumption of the battery before the next charge. In this section, we are interested in the characteristics of fluid level, such as the probability density function, the expected value, and the LST of the fluid level [22]. Let $\{\mathcal{C}(t)\}_{t\geq 0}$ be the fluid level at time t, and let $\mathcal{C}$ be the steady-state random variable of $\{\mathcal{C}(t)\}_{t\geq 0}$ as t goes to ∞. Figure 3.1 demonstrates a typical sample path of the fluid level and the virtual waiting time of the driving $M/G/1$ queue. As observed in the Figure 3.1, once the fluid level approaches level 0^+, the leaking rate of the fluid queue approaches 0^+, but never reaches to 0. This observation in Figure 3.1 hints that the point mass of the fluid level at zero is zero.

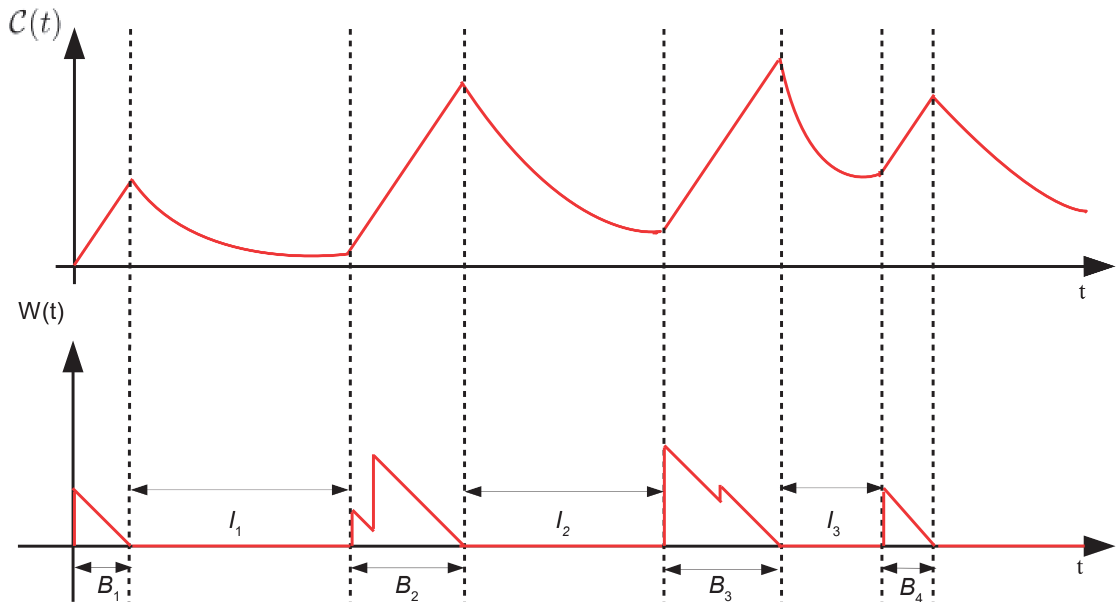

Figure 3.1: A typical sample path of the fluid queue with leaking rate depending on fluid level and the virtual waiting time of the driving queue. Top corresponds to the fluid queue. Bottom corresponds to the driving $M/G/1$ queue.

3.1.2 Level-crossing equation

As usual, denote by $U_t(x)$ and $D_t(x)$ the number of upcrossings and downcrossings of level x during $(0, t)$, respectively. Using an LC argument [12, Theorem 6.1, p. 304], we obtain

$$\lim_{t\to\infty} \frac{U_t(x)}{t} = k_1 f_1(x) \quad \text{and} \quad \lim_{t\to\infty} \frac{D_t(x)}{t} = k_2 \cdot x \cdot f_2(x), \quad x > 0, \tag{3.1}$$

and

$$f(x) = f_1(x) + f_2(x), \quad x > 0, \tag{3.2}$$

where $\lim\limits_{t\to\infty} U_t(x)/t$ and $\lim\limits_{t\to\infty} D_t(x)/t$ are the upcrossing and downcrossing rates of level x, $f_1(x)$ and $f_2(x)$ are the partial pdfs of the fluid level defined in similar manner as in (2.5), and $f(x)$ is the total pdf of the fluid level. Balancing the upcrossing and downcrossing rates of level x yields

$$k_1 f_1(x) = k_2 \cdot x \cdot f_2(x) \implies f_1(x) = \frac{k_2}{k_1} x f_2(x), \quad x > 0. \tag{3.3}$$

Substituting (3.3) into (3.2), we have

$$f_1(x) = \frac{k_2 x}{k_1 + k_2 x} f(x) \quad \text{and} \quad f_2(x) = \frac{k_1}{k_1 + k_2 x} f(x). \tag{3.4}$$

Substituting (3.4) into (3.1) agrees with

$$\lim_{t\to\infty} \frac{U_t(x)}{t} = \lim_{t\to\infty} \frac{D_t(x)}{t}, \quad x > 0. \tag{3.5}$$

The expression (3.5) is a check on a basic rate balance property of the LC method. In addition, by the LC argument, we have

$$k_1 f_1(0^+) = \lambda \pi_{\mathcal{E}} = k_2 \cdot 0^+ \cdot f_2(0^+), \tag{3.6}$$

where $\pi_{\mathcal{E}}$ is defined as point mass of the fluid level at level 0. From second equality in (3.6), we have

$$\pi_{\mathcal{E}} = 0^+ \cdot k_2 \cdot f_2(0^+)/\lambda, \tag{3.7}$$

which implies that $\pi_{\mathcal{E}} = 0$ for this particular fluid queue. Substituting $f_2(x)$ in (3.4) into $\lim\limits_{t\to\infty} D_t(x)/t$ in (3.1) yields

$$\lim_{t\to\infty} \frac{D_t(x)}{t} = \frac{k_1 k_2 x}{k_1 + k_2 x} f(x), \quad x > 0. \tag{3.8}$$

Remark 3.1.1. *The unit for k_2 [1/time] is different than the unit of the net input rate k_1 [gram/time], since the unit of fluid, x, is grams, and the units for the leaking rate,*

$k_2 \cdot x$, *is [gram/time].*

Alternative expression of upcrossing rate of level x in the fluid queue: The alternative expression of the rate at which the sample path of fluid level upcrosses level x is

$$\lim_{t\to\infty} \frac{U_t(x)}{t} = \lambda \int_{0^+}^{\infty} \overline{B}\left(\frac{x-y}{k_1}\right) f_2(y)dy, \tag{3.9}$$

where $\overline{B}(\bullet)$ is the ccdf of the busy period of the driving $M/G/1$ queue. The expression $\overline{B}((x-y)/k_1)$ in (3.9) accounts for the fact that the time required for the sample path to upcross level x starting from level y in the interval $(0, x)$ is $(x-y)/k_1$. Substituting $f_2(x)$ in terms of $f_1(x)$ expressed in (3.3) into (3.9), and balancing two different expressions of upcrossing rate of level x, (3.1) and (3.9), yields

$$f_1(x) = \lambda \int_{0^+}^{\infty} \frac{1}{k_2 y} \cdot \overline{B}\left(\frac{x-y}{k_1}\right) \cdot f_1(y)dy. \tag{3.10}$$

Alternative expression of downcrossing rate of level x in the fluid queue: Using the basic property of the LC method, namely: rate in = rate out, the alternative expression of the rate at which the sample path of fluid level downcrosses level x is

$$\lim_{t\to\infty} \frac{D_t(x)}{t} = \lambda \int_{0^+}^{\infty} \overline{B}\left(\frac{x-y}{k_1}\right) f_2(y)dy, \tag{3.11}$$

Balancing two different expression of downcrossing rate of level x, (3.1) and (3.11), yields

$$f_2(x) = \frac{\lambda}{k_2 x} \int_{0^+}^{\infty} \overline{B}\left(\frac{x-y}{k_1}\right) \cdot f_2(y)dy. \tag{3.12}$$

Summing (3.10) and (3.12) gives

$$f(x) = \frac{\lambda(k_1 + k_2 x)}{k_2 x} \int_{y=0^+}^{x} \overline{B}\left(\frac{x-y}{k_1}\right) \cdot \frac{1}{k_1 + k_2 y} \cdot f(y)dy. \tag{3.13}$$

To further support the expression $f(x)$ in (3.13) is valid, we use different approach (method of the 'page' by Brill [12, Chapter 4, pp. 162 - 254, and Section 10.10, pp. 433 - 439]) to find the $f(x)$. The details of the *different approach* can be found in Appendix A.3. Here, to increase the confidence of the expression (3.13) is valid, we evaluate the unit of the both sides of (3.13), namely

$$\left[\frac{1}{gram}\right] = \left[\frac{\frac{1}{T} \cdot \left(\frac{gram}{T} + \frac{gram}{T}\right)}{\frac{gram}{T}}\right] \cdot \left[1 \cdot \frac{T}{gram} \cdot \frac{1}{gram} \cdot gram\right] = \left[\frac{1}{gram}\right]. \tag{3.14}$$

Solving the functional equation of the form in (3.13) is challenge, instead we solve for $f(x)$ using the expression (3.4) and the functional equation for $f_2(x)$ expressed in (3.12).

3.1.3 Probability distribution of fluid level

In this section, we express the pdf of the fluid level in terms of the pdf $f_2(x)$ defined in (3.4). First, we summarize the findings of the pdf $f_2(x)$ in Section 3.1.2. For $x > 0$, by using the expressions (3.4), the pdf of the fluid level $f(x)$ can be expressed in terms of $f_2(x)$:

$$f(x) = \left(1 + \frac{k_2 x}{k_1}\right) f_2(x), \quad x > 0, \tag{3.15}$$

where $f(x)$ is the total pdf of the fluid level, and $f_2(x) > 0$ is the partial pdf defined as

$$f_2(x) = \frac{d}{dx} \lim_{t \to \infty} \mathbb{P}[\mathcal{C}(t) \leq x, Z(t) = 2], \tag{3.16}$$

where $\{Z(t)\}_{t\geq 0}$ is defined in (2.2). The functional equation form for $f_2(x)$ is expressed in (3.12). More importantly, as observed in the expression (3.16), it is clearly that $f_2(x)$ is not a well-defined pdf such that

$$\int_{0^+}^{\infty} f_2(x)dx \neq 1, \tag{3.17}$$

and equivalently

$$\lim_{s\to 0} \mathcal{L}_{f_2}(s) \neq 1. \tag{3.18}$$

Denote by $\{\mathcal{C}(t), Z(t)\}_{t\geq 0}$ the process of the fluid queue. To find the pdf of the fluid level, we decompose the process $\{\mathcal{C}(t), Z(t)\}_{t\geq 0}$ into two sub-processes corresponding to $\{Z(t) = 1\}_{t\geq 0}$ and $\{Z(t) = 2\}_{t\geq 0}$, namely:

$$\{\mathcal{C}(t), Z(t)\}_{t\geq 0} = \{\mathcal{C}(t), Z(t) = 1\}_{t\geq 0} \bigcup \{\mathcal{C}(t), Z(t) = 2\}_{t\geq 0}. \tag{3.19}$$

We refer to the page that corresponds to the process $\{\mathcal{C}(t), Z(t)\}_{t\geq 0}$ as the cover page. Likewise, we refer to the page that corresponds to the process $\{\mathcal{C}(t), Z(t) = 1\}_{t\geq 0}$ as the page 1, and the page that corresponds to the process $\{\mathcal{C}(t), Z(t) = 2\}_{t\geq 0}$ as the page 2. The sample path of the process $\{Z(t)\}_{t\geq 0}$, and the sample paths of the fluid level on page 1 and 2, and cover page are illustrated in Figure 3.2. As observed in the figures, the fluid enters the fluid content is governed by the alternative renewal process $\{Z(t)\}_{t\geq 0}$. Denote by B the busy period of the driving $M/G/1$ queue and by I the idle period of the driving $M/G/1$ queue, we have

$$\lim_{t\to\infty} \mathbb{P}[Z(t) = 1] = \frac{\mathbb{E}[B]}{\mathbb{E}[B] + \mathbb{E}[I]} \quad \text{and} \quad \lim_{t\to\infty} \mathbb{P}[Z(t) = 2] = \frac{\mathbb{E}[I]}{\mathbb{E}[B] + \mathbb{E}[I]}. \tag{3.20}$$

Solution for $f_2(x)$***:*** Let $\mathcal{C}_2$ denote the steady-state random variable of $\{\mathcal{C}(t), Z(t) =$

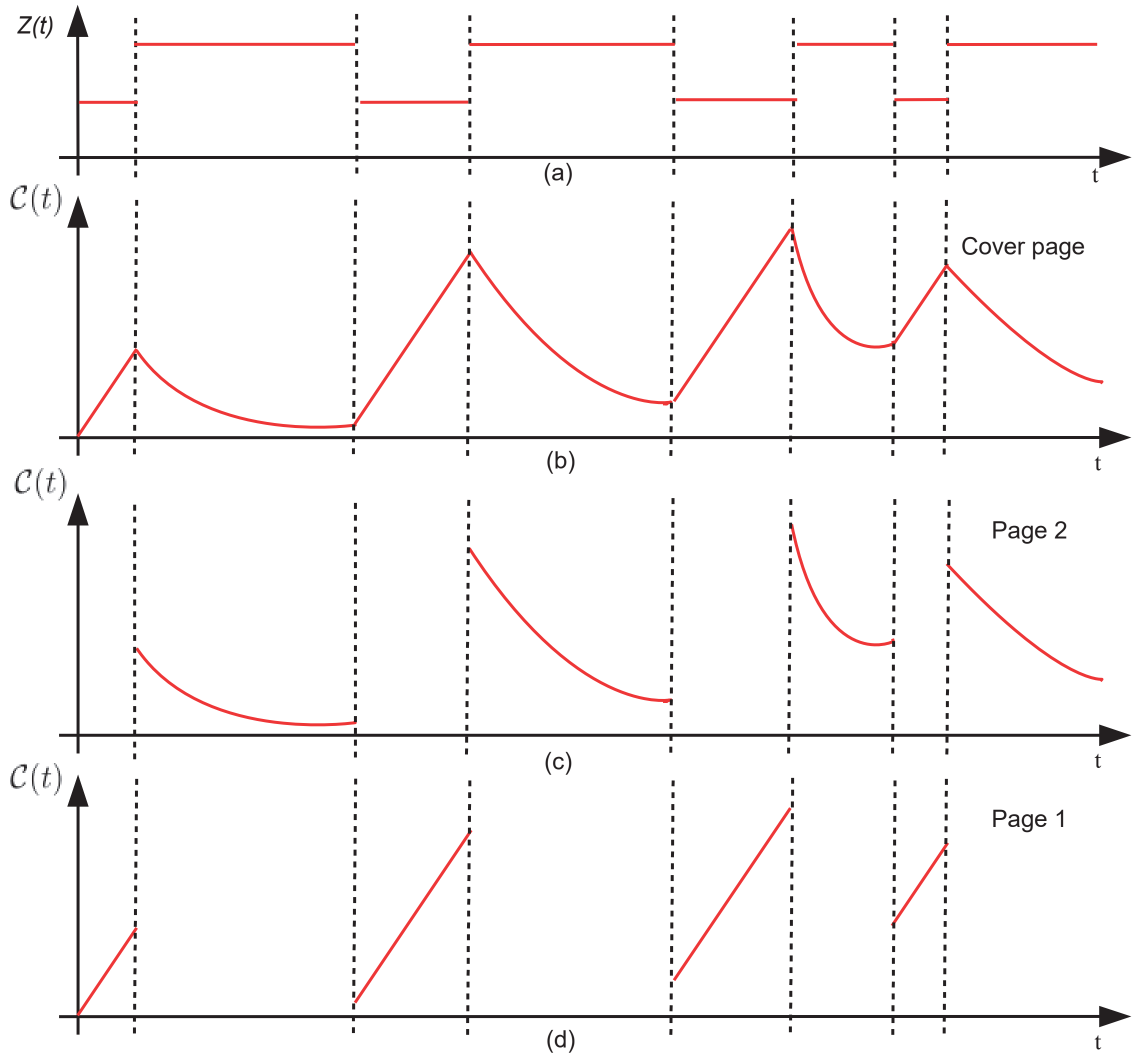

Figure 3.2: (a) a sample path of $\{Z(t)\}_{t\geq 0}$, (b) a sample path of $\{\mathcal{C}(t), Z(t)\}_{t\geq 0}$, (c) a sample path of $\{\mathcal{C}(t), Z(t) = 2\}_{t\geq 0}$, and (d) a sample path of $\{\mathcal{C}(t), Z(t) = 1\}_{t\geq 0}$.

$2\}_{t\geq 0}$ as t goes to ∞. For $s > 0$, define

$$\widetilde{\mathcal{C}}_2(s) = \int_{0^-}^{\infty} e^{-sx} d\mathbb{P}[\mathcal{C}_2 \leq x], \quad \text{and} \quad \widetilde{B}(s) = \int_{0^+}^{\infty} e^{-sx} d\mathbb{P}[B \leq x], \tag{3.21}$$

as the LST of the pdf of the fluid level and the LST of the pdf of the busy period of the driving $M/G/1$ queue respectively. Multiplying e^{-xs} on both sides of (3.12) and integrating with respect to x over $(0, \infty)$ yields

$$\int_{0^+}^{\infty} e^{-sx} x f_2(x) dx = \frac{\lambda}{k_2} \int_{x=0^+}^{\infty} \int_{y=0^+}^{x} e^{-xs} \overline{B}\left(\frac{x-y}{k_1}\right) f_2(y) dy dx, \quad x > 0. \tag{3.22}$$

In the double integral, replacing e^{-xs} by $e^{-(x-y+y)s}$, and interchanging limits yields

$$\begin{aligned}\int_{x=0^+}^{\infty} \int_{y=0^+}^{x} & e^{-(x-y+y)s} \overline{B}\left(\frac{x-y}{k_1}\right) f(y) dy dx \\ &= \int_{y=0^+}^{\infty} e^{-ys} f_2(y) \int_{x=0^+}^{\infty} e^{-(x-y)s} \overline{B}\left(\frac{x-y}{k_1}\right) dx dy.\end{aligned}$$

Letting $u = (x-y)/k_1$ and $k_1 du = dx$, and substituting into the above expression yields

$$\int_{y=0^+}^{\infty} e^{-ys} f_2(y) \int_{x=0^+}^{\infty} e^{-(x-y)s} \overline{B}\left(\frac{x-y}{k_1}\right) dx dy$$

$$= k_1 \int_{x=0^+}^{\infty} e^{-xs} d\mathbb{P}[\mathcal{C}_2 \leq x] \int_{u=0^+}^{\infty} e^{-uk_1 s} \overline{B}(u) du. \tag{3.23}$$

Using a property of LST[1] and substituting (3.23) into (3.22) yield

$$\frac{\widetilde{\mathcal{C}_2}'(s)}{\widetilde{\mathcal{C}_2}(s)} = -\lambda \frac{1 - \widetilde{B}(k_1 s)}{k_2 s}, \quad s > 0, \tag{3.24}$$

with solution

$$\widetilde{\mathcal{C}_2}(s) = C_0 \exp\left(-\frac{\lambda}{k_2} \int_0^s \frac{1 - \widetilde{B}(k_1 w)}{w} dw\right), \quad s > 0, \tag{3.25}$$

where C_0 is an unknown. As highlight in the (3.18), letting $s \to 0$ in (3.25) do not give 1, hence we cannot find C_0 using this property of LST for pdf.

Denote by $\mathcal{C}_{x|2}$ the steady-state random variable of $\{\mathcal{C}(t)|Z(t) = 2\}_{t \geq 0}$ as t goes to ∞,

[1]For $x > 0$, $\mathcal{L}(xf(x)) = -\widetilde{f}'(s)$ where $\widetilde{f}(s)$ is the LST of $f(x)$.

and by $g(x)$ the pdf of the random variable $\mathcal{C}_{x|2}$. The cdf of $\mathcal{C}_{x|2}$ is

$$\mathbb{P}[\mathcal{C}_{x|2} \le x] = \frac{\mathbb{P}[\mathcal{C} \le x, Z = 2]}{\mathbb{P}[Z = 2]} = \frac{\mathbb{P}[\mathcal{C}_2 \le x]}{\mathbb{P}[Z = 2]}, \quad x > 0, \tag{3.26}$$

where $\mathbb{P}[Z = 2]$ is defined in (3.20). The expression (3.26) suggests that

$$g(x) = \frac{1}{\mathbb{P}[Z = 2]} f_2(x), \quad \text{and} \quad \mathcal{L}_g(s) = \frac{1}{\mathbb{P}[Z = 2]}, \widetilde{\mathcal{C}}_2(s). \tag{3.27}$$

where $\widetilde{\mathcal{C}}_2(s)$ is the LST of the $f_2(x)$ defined in (3.25). Letting s go to 0 in (3.27) yield $C_0 = \mathbb{P}[Z = 2]$. Substituting $f_2(x) = \mathbb{P}[Z = 2] \cdot g(x)$ into (3.15) we have,

$$f(x) = \left(1 + \frac{k_2 x}{k_1}\right) \cdot \mathbb{P}[Z = 2] \cdot g(x), \quad x > 0. \tag{3.28}$$

Taking LST in both sides of (3.28), we have[2]

$$\widetilde{\mathcal{C}}(s) = \left(\widetilde{\mathcal{C}}_2(s) - \frac{k_2}{k_1} \widetilde{\mathcal{C}}_2'(s)\right), \tag{3.29}$$

Driving process: $M/M/1$ ***queue:*** Assume that the driving queue of the fluid queue is an $M/M/1$ queue with arrival rate λ and service rate μ such that $\lambda < \mu$. The LST of the pdf of the busy period is well-known (see [45, eq. (31), p. 390] for the details):

$$\widetilde{B}(s) = \frac{\mu + \lambda + s - \sqrt{(\mu + \lambda + s)^2 - 4\lambda\mu}}{2\lambda}, \quad s > 0. \tag{3.30}$$

Fixing $\lambda = 1$ and $\mu = 2$, and substituting (3.30) into (3.29), Figure 3.3 illustrates the LST of the pdf of the fluid level for 4 cases[3]: $k_1 = 6, k_2 = 2; k_1 = 1, k_2 = 3$, and $k_1 = 1, k_2 = 4$, and $k_1 = 1, k_2 = 5$.

[2] For $x > 0$, $\mathcal{L}(xf(x)) = -\widetilde{f}'(s)$ where $\widetilde{f}(s)$ is the LST of $f(x)$.

[3] The pdf and the cdf of the fluid level can be numerically computed using Euler and Post-Widder algorithm. The details of the numerical inversion of LST of probability distributions and the algorithms are discussed in [2].

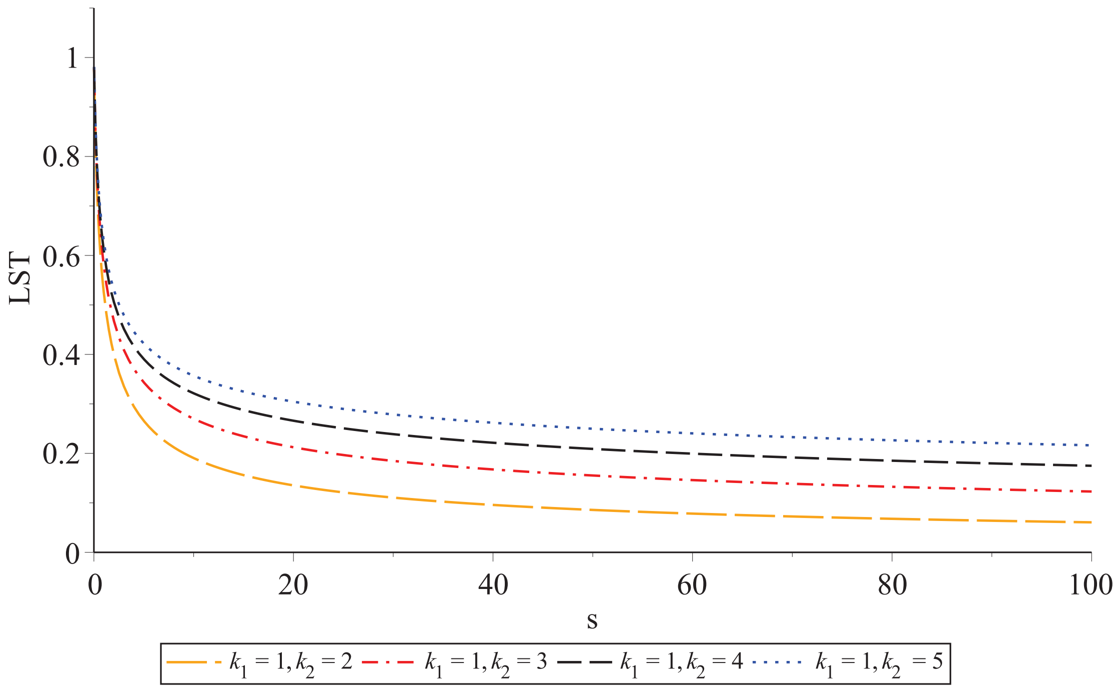

Figure 3.3: Laplace-Stieltjes transform of the pdf of the fluid level. Driving queue is $M/M/1$.

3.1.4 Mapping the integral equation of the probability density function of the workload of $M/M/r(\cdot)$ and the fluid level

Consider an $M/M/r(\cdot)$ dam model (see [12, Section 10.8, pp.423 - 426] for the details of the model and setup of the LC equation) with releasing rate depending on the level of the workload, namely: k_2x, and arrival rate λ and service rate μ. Denote by $D(t)$ the workload at time t for the $M/M/r(\cdot)$. A typical sample path of the $\{D(t)\}_{t\geq 0}$ is illustrated in Figure 3.4. Brill [12, Section 10.8.2, eq. (10.57), pp. 424 - 426] found that the LC equation of the pdf of the workload of the $M/M/r(\cdot)$ dam is

$$h(x) = \frac{\lambda}{k_2x}\int_{y=0^+}^{x} e^{-\mu(x-y)}g(y)dy, \quad x > 0, \tag{3.31}$$

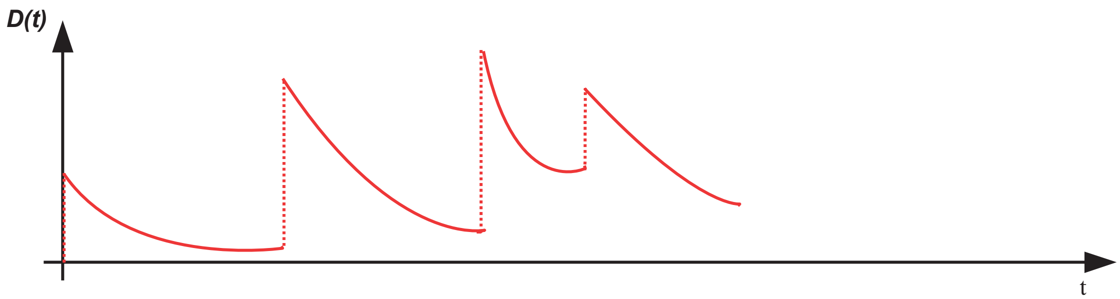

Figure 3.4: A sample path of $\{D(t)\}_{t\geq 0}$.

with solution that $h(x)$ is a Gamma pdf, namely (see Brill [12, eq. (6.44), p. 317] for details)

$$h(x) = \frac{1}{\Gamma(\lambda/k_2)}\mu(\mu x)^{(\lambda/k-1)}\exp(-\mu x), \quad x > 0, \tag{3.32}$$

with

$$\mathcal{L}_h(s) = \left(1 + \frac{s}{\mu}\right)^{-\lambda/k_2}, \quad s > 0, \tag{3.33}$$

where $\mathcal{L}_h(s)$ is the LST of the pdf $h(x)$. The goal of this section is expressing the $h(x)$ in terms of the pdf of the fluid level $f_2(x)$.

To express $h(s)$ in terms of $f_2(x)$, we fix the *net* input rate $k_1 = 1$ and assume that B is exponential distributed with rate μ in this section. A typical sample path of the $\{\mathcal{C}(t)\}_{t\geq 0}$ is illustrated in Figure 3.5 (a). To connect the sample paths of the $\{D(t)\}_{t\geq 0}$ and the $\{\mathcal{C}(t)\}_{t\geq 0}$, we focus on the sample path of the fluid level on 'page 2', i.e. the fluid process $\{\mathcal{C}(t), Z(t) = 2\}_{t\geq 0}$. The sample path of $\{\mathcal{C}(t), Z(t) = 2\}_{t\geq 0}$ is illustrated in Figure 3.5 (b). The steady-state partial pdf of the sub-process $\{\mathcal{C}(t), Z(t) = 2\}_{t\geq 0}$ as $t \to \infty$ is expressed in (3.4), namely

$$f_2(x) = \left(\frac{1}{1 + k_2 x}\right) f(x), \quad x > 0. \tag{3.34}$$

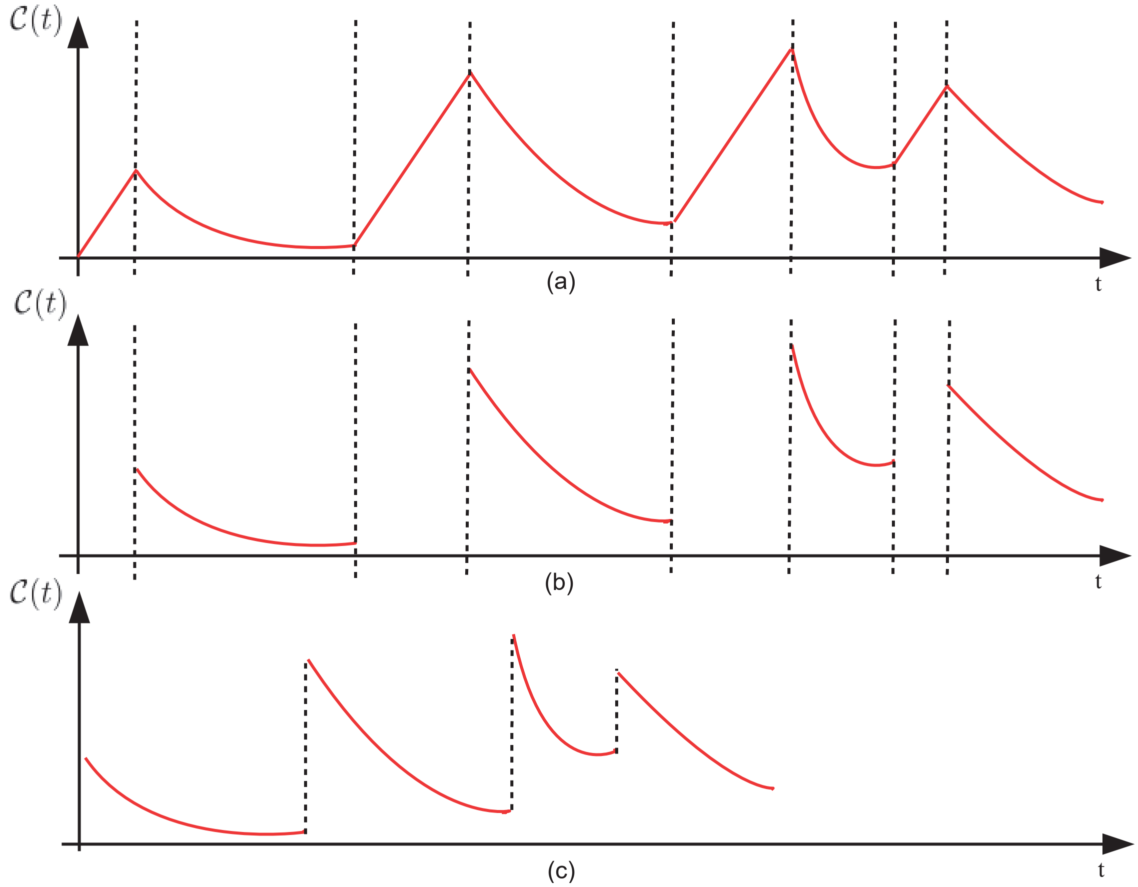

Figure 3.5: (a) a sample path of $\{\mathcal{C}(t)\}_{t\geq 0}$, (b) a sample path of $\{\mathcal{C}(t), Z(t) = 2\}_{t\geq 0}$, (c) a sample path of $\{\mathcal{C}(t) \mid Z(t) = 2\}_{t\geq 0}$.

Consider the process $\{\mathcal{C}(t) \mid Z(t) = 2\}_{t\geq 0}$ which illustrated in Figure 3.5 (c). Comparing the sample path of $\{\mathcal{C}(t) \mid Z(t) = 2\}_{t\geq 0}$ in Figure 3.5 (c) and the sample path of $\{D(t)\}_{t\geq 0}$ in Figure 3.4, these two sample paths share the same characteristics. This suggests that the pdf associated with the process $\{D(t)\}_{t\geq 0}$ is identical to the pdf associated with the process $\{\mathcal{C}(t)|Z(t) = 2\}_{t\geq 0}$. Thus, we can express $h(x)$ in terms of $f_2(x)$, namely

$$h(x) = \frac{d}{dx}\left(\frac{\mathbb{P}[\mathcal{C} \leq x, Z = 2]}{\mathbb{P}[Z = 2]}\right) = \left(1 + \frac{\lambda}{\mu}\right) f_2(x), \tag{3.35}$$

where $\lim_{t\to\infty} P[Z(t) = 2] = \mu/\ (\mu + \lambda)$, since we assume that B is exponential distributed with rate μ. To validate the expression in (3.16)), we take LST on both sides (3.35).

This yields

$$\mathcal{L}_h(s) = \left(1 + \frac{\lambda}{\mu}\right) \mathcal{L}_{f_2}(s). \tag{3.36}$$

To evaluate $(1 + \lambda/\mu)\mathcal{L}_{f_2}(s)$, we substitute $\left(1 - \widetilde{B}(w)\right) = \mu/(\mu + w)$ into (3.25). This gives

$$\left(1 + \frac{\lambda}{\mu}\right) \mathcal{L}_{f_2}(s) = \exp\left(-\frac{\lambda}{k_2} \int_0^s \frac{1}{\mu + w} dw\right) = \left(1 + \frac{s}{\mu}\right)^{-\lambda/k_2}, \tag{3.37}$$

which suggests that $(1 + \lambda/\mu)f_2(x)$ is Gamma distributed. Comparing the expressions (3.35) and (3.33), we confirm that the expression (3.35) is valid.

3.2 Fluid queue with different leaking rates

3.2.1 Introduction

We consider a fluid queue with infinite capacity as discussed in Chapter 2 such that the activity, silence, and empty periods of the fluid queue are modulated by an $M/G/1$ queue (hereafter modified fluid queue). During the busy period of the driving $M/G/1$ queue, the fluid content fills at *net* rate r_1, whereas during the idle period of the driving queue, the fluid content empties at one of two different possible rates decided by a system manager at the time point that the activity period of the fluid queue starts at the fluid queue. With probability p_2, the fluid content empties at rate r_2, and with probability p_3, the fluid content empties at rate r_3, such that $p_2 + p_3 = 1$. In addition, once the *continuous* leaking rate is determined, the leaking rate stays the same for the entire busy cycle of the driving queue. Figure 3.6 demonstrates a typical sample path of the modified fluid queue and the driving $M/G/1$ queue. As observed in Figure 3.6, the $M/G/1$ queue

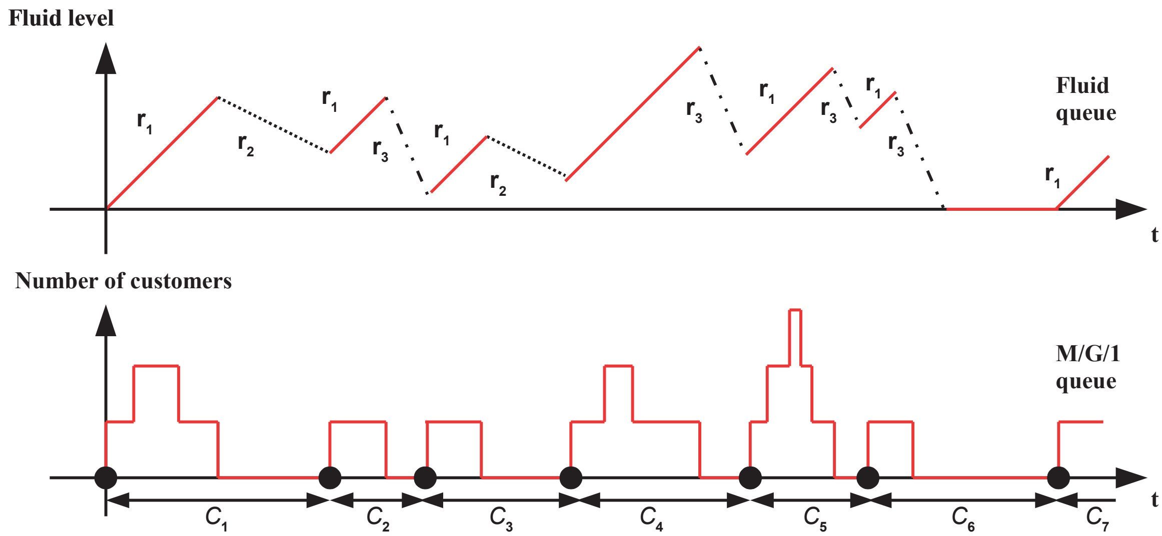

Figure 3.6: A sample path of the fluid queue and the driving $M/G/1$ queue

consists of six completed busy cycles C_i and one incompleted busy cycle. During the 1^{st} and 3^{rd} cycles, the fluid content empties at rate r_2, whereas during the 2^{nd}, 4^{th} 5^{th}, and 6^{th} cycles, the fluid content empties at rate r_3. More importantly, the leaking rate of the fluid queue for each busy cycle of the driving queue is determined by the system manager at the time points (which is indicated by circles in Figure 3.6) when arrivals arrive to the empty $M/G/1$ queue.

3.2.2 Level-crossing equation

Denote by $\mathcal{C}(t)$, $Z(t)$, and $\mathcal{R}(t)$ the fluid level, the states of the driving queue defined in (2.2), and the *net* input rate of the fluid queue at time $t > 0$. Denote by $\mathbb{F}_{i,j,t}(x)$ the partial cdf's of fluid content at time t,

$$\mathbb{F}_{i,j,t}(x) := \mathbb{P}[\mathcal{C}(t) \leq x, Z(t) = i, \mathcal{R}(t) = r_j], \quad i = 1, 2; \; j = 1, 2, 3; \; x > 0, \tag{3.38}$$

and by $f_{i,j}(x)$ the partial steady-state pdfs corresponding to $\mathbb{F}_{i,j,t}(x)$ as $t \to \infty$,

$$f_{i,j}(x) := \frac{d}{dx} \lim_{t\to\infty} \mathbb{P}[\mathcal{C}(t) \leq x, Z(t) = i, \mathcal{R}(t) = r_j], \quad i = 1, 2;\ j = 1, 2, 3,\ x > 0. \quad (3.39)$$

The marginal steady-state pdf of the fluid content is

$$f(x) = f_{1,1}(x) + f_{2,2}(x) + f_{2,3}(x), \quad x > 0. \quad (3.40)$$

Denote by $\mathbb{F}_{i,j}(x)$ the steady-state cdf of $\mathbb{F}_{i,j,t}(x)$ as $t \to \infty$. Then we have

$$\mathbb{F}_{i,j}(x) = \pi_{\mathcal{E}} + \int_{x=0^+}^{\infty} f_{i,j}(x)dx, \quad (3.41)$$

where $\pi_{\mathcal{E}}$ is the point mass of the fluid level at 0. To simplify the notation in this section, we replaced $f_{i,j}(x)$ by $f_j(x)$. Note that the pdfs $f(x)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$ are defined only when $x > 0$.

To find the steady-state distribution of the fluid level, we use the page method [12, Chapter 4, pp. 162 - 254, and Section 10.10, pp. 433 - 439]. The page method allows researchers to write a total pdf of a process in terms of its sub-process pdfs. Thus, to apply the page method, first we decompose the stochastic process $\{\mathcal{C}(t), Z(t), \mathcal{R}(t)\}_{t\geq 0}$ into 4 different sub-processes and record the sample path of each process on 4 different 'pages'. The sample path on page 1 corresponds to the process $\{\mathcal{C}(t), Z(t) = 1, \mathcal{R}(t) = r_1\}_{t\geq 0}$, the sample path on page 2 corresponds to the process $\{\mathcal{C}(t), Z(t) = 2, \mathcal{R}(t) = r_2\}_{t\geq 0}$, the sample path on page 3 corresponds to the process $\{\mathcal{C}(t), Z(t) = 2, \mathcal{R}(t) = r_3\}_{t\geq 0}$, and the sample path on page 4 corresponds to the process $\{\mathcal{C}(t) = 0, Z(t) = 2, \mathcal{R}(t) = 0\}_{t\geq 0}$. Figure 3.7 illustrates a typical sample path of the $\{\mathcal{C}(t), Z(t), \mathcal{R}(t)\}_{t\geq 0}$ and the sub-processes of it. Finally, we refer to the page that corresponds to the original process $\{\mathcal{C}(t), Z(t), \mathcal{R}(t)\}_{t\geq 0}$ as the cover page (note that in LC, 'pages' are also called 'sheets').

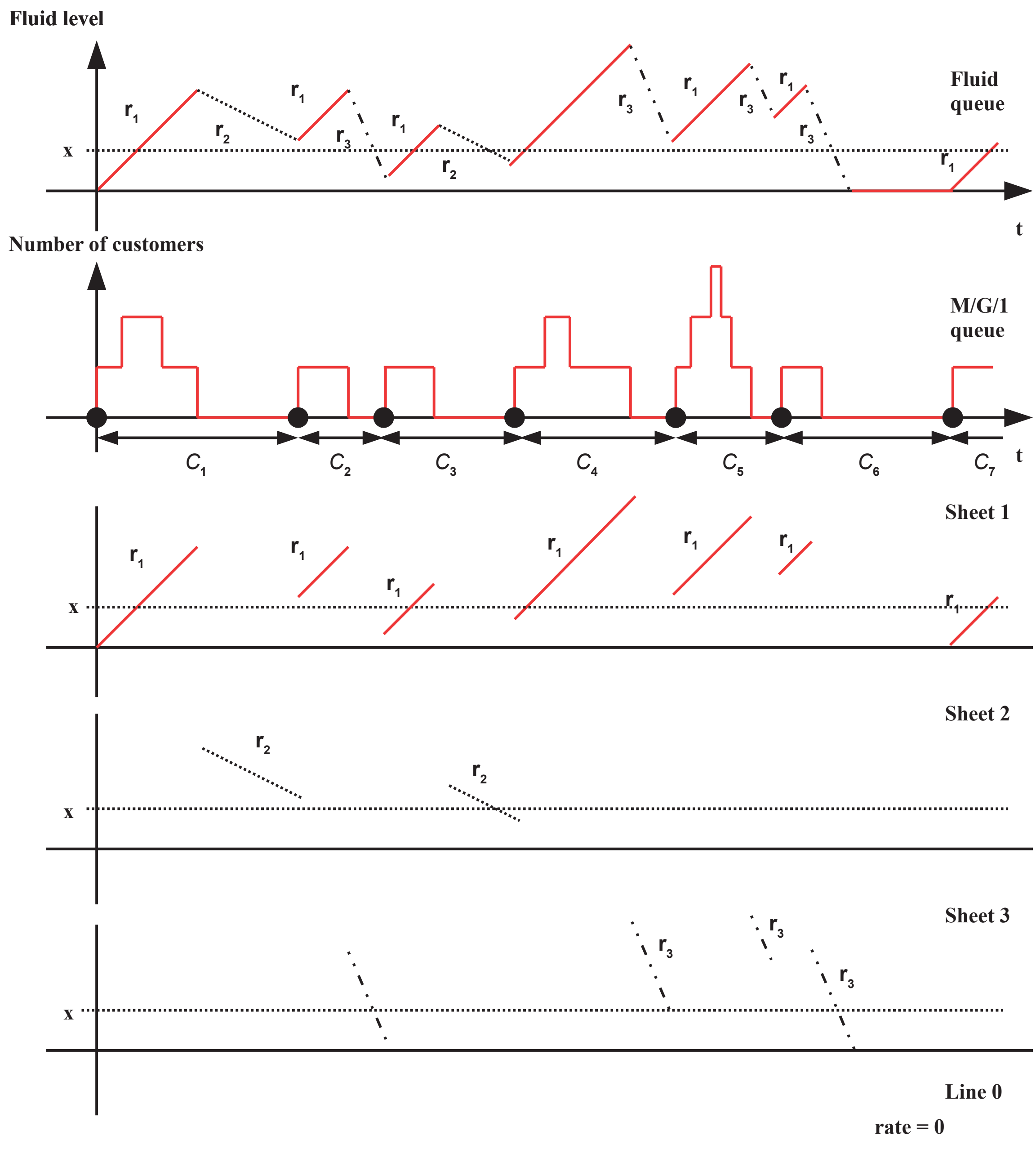

Figure 3.7: Sheet and line 0 diagram

Page equations:

Page 1: Balancing the rate at which the sample path enters the composition state $((x, \infty), 1)$, $x > 0$, and leaves the composition state $((x, \infty), 1)$, $x > 0$, where 1 corresponds to the page 1, gives us

$$
\begin{aligned}
r_1 f_1(x) + \lambda \int_{y=x}^{\infty} f_2(y)dy + \lambda \int_{y=x}^{\infty} f_3(y)dy \\
&= \lambda \pi_{\mathcal{E}} \overline{B}\left(\frac{x}{r_1}\right) \\
&+ \lambda \int_{y=x}^{\infty} f_2(y)dy + \lambda \int_{y=0^+}^{x} \overline{B}\left(\frac{x-y}{r_1}\right) f_2(y)dy \\
&+ \lambda \int_{y=x}^{\infty} f_3(y)dy + \lambda \int_{y=0^+}^{x} \overline{B}\left(\frac{x-y}{r_1}\right) f_3(y)dy,
\end{aligned} \tag{3.42}
$$

where $\overline{B}(\cdot)$ is ccdf of the busy period of the driving $M/G/1$ queue. The terms on the left-hand side of (3.42) correspond to the rate at which the sample path enters $((x,\infty),1)$, namely: (i) the sample path upcrosses level x at rate $r_1 f_1(x)$; (ii) the rate at which the sample path enters $((x,\infty),1)$ from $((x,\infty),2)$; (iii) the rate at which the sample path enters $((x,\infty),1)$ from $((x,\infty),3)$. The terms on the right-hand side correspond to the rate at which the sample path leaves $((x,\infty),1)$ on page 1, namely: (i) the rate at which the sample path enters $((x,\infty),1)$ from level 0 and the busy period of the driving queue ends via the fluid level above level x; (ii) the rate at which the sample path enters $((x,\infty),1)$ from $((x,\infty),2)$; (iii) the rate at which the sample path enters $((x,\infty),1)$ from $((0^+,x),2)$ and the activity period of the fluid queue ends at level y in (x,∞) on page 1; (iv) the rate at which the sample path enters $((x,\infty),1)$ from $((x,\infty),3)$; (v) the rate at which the sample path enters $((x,\infty),1)$ from $((0^+,x),3)$ and the activity period of the fluid queue ends at level y in (x,∞) on page 1.

To further support the level crossing equation (3.42), we examine the unit of the left-hand and right-hand sides of (3.42). Let T be the unit of time and *grams* be the unit of the fluid. The unit of the right-hand side of (3.42) is

$$
\left[r_1 f_1(x) + \lambda \int_{y=x}^{\infty} f_2(y)dy + \lambda \int_{y=x}^{\infty} f_3(y)dy \right] = \left[\frac{grams}{T} \cdot \frac{1}{grams} + \frac{1}{T} \right] = \left[\frac{1}{T} \right]. \tag{3.43}
$$

The unit of the first term on the right-hand side of (3.42) is

$$\left[\lambda\pi_{\mathcal{E}}\overline{B}\left(\frac{x}{r_1}\right)\right] = \left[\frac{1}{T}\right], \tag{3.44}$$

and the units of the second and third terms on the right-hand side of (3.42) are

$$\left[\int_{y=x}^{\infty} f_2(y)dy + \lambda\int_{z=0^+}^{x} \overline{B}\left(\frac{x-y}{r_1}\right) f_2(y)dy\right] = \left[\frac{1}{T}\cdot\frac{1}{grams}\cdot grams\right] = \left[\frac{1}{T}\right]. \tag{3.45}$$

Similarly, the units of the remaining terms on the right-hand side of (3.42) are $[1/T]$ as well. Thus, the units of the left- and right-hand side of (3.42) are equal.

Remark 3.2.1. *The unit of the left- and right-hand side in the level crossing equation being equal does not guarantee that the level crossing equation is defined correctly. Unit checking can only be used to verify that the level crossing equation is not defined incorrectly, i.e. it is a necessary condition for the level crossing equation to be correct, but it is not a sufficient condition.*

Next, simplifying (3.42) gives us,

$$r_1 f_1(x) = \lambda\pi_{\mathcal{E}}\overline{B}\left(\frac{x}{r_1}\right) + \lambda\int_{y=0}^{x} \overline{B}\left(\frac{x-y}{r_1}\right)(f_2(y)+f_3(y))\,dy, \quad x>0. \tag{3.46}$$

Both sides of (3.46) can be interpreted as the rate at which the sample path upcrosses level x.

Pages 2 and 3: Balancing the rate at which the sample path enters and leaves $((x,\infty),2)$, $x >$

0, and the rate at which the sample path enters and leaves $((x,\infty),3), x>0$, yields

$$
\begin{aligned}
p_2\lambda\pi_{\mathcal{E}}\overline{B}\left(\frac{x}{r_1}\right) &+ p_2\lambda\int_{z=0}^{x}\overline{B}\left(\frac{x-y}{r_1}\right)f_2(y)dy + p_2\lambda\int_{y=x}^{\infty}f_2(y)dy \\
&+ p_2\lambda\int_{z=0}^{x}\overline{B}\left(\frac{x-y}{r_1}\right)f_3(y)dy + p_2\lambda\int_{y=x}^{\infty}f_3(y)dy \\
&= r_2f_2(x) + \lambda\int_{y=x}^{\infty}f_2(y)dy, \qquad (3.47)
\end{aligned}
$$

and

$$
\begin{aligned}
p_3\lambda\pi_{\mathcal{E}}\overline{B}\left(\frac{x}{r_1}\right) &+ p_3\lambda\int_{z=0}^{x}\overline{B}\left(\frac{x-y}{r_1}\right)f_2(y)dy + p_3\lambda\int_{y=x}^{\infty}f_2(y)dy \\
&+ p_3\lambda\int_{z=0}^{x}\overline{B}\left(\frac{x-y}{r_1}\right)f_3(y)dy + p_3\lambda\int_{y=x}^{\infty}f_3(y)dy \\
&= r_3f_3(x) + \lambda\int_{y=x}^{\infty}f_3(y)dy, \qquad (3.48)
\end{aligned}
$$

respectively. The interpretations of (3.47) and (3.48) are similar to (3.42). Addition of (3.42), (3.47), and (3.48) yields

$$r_1f_1(x) = r_2f_2(x) + r_3f_3(x), \quad x>0. \qquad (3.49)$$

Balancing the rate at which the sample path enters and leaves level 0, we have

$$r_1f_1(0^+) = r_2f_2(0^+) + r_3f_3(0^+) = \lambda\pi_{\mathcal{E}}, \qquad (3.50)$$

$$r_2f_2(0^+) = p_2\lambda\pi_{\mathcal{E}} \quad \text{and} \quad r_2f_3(0^+) = p_3\lambda\pi_{\mathcal{E}}. \qquad (3.51)$$

The expression (3.49) can be achieved by recognizing the following property, namely,

$$\lim_{t\to\infty}\frac{U_t^1(x)}{t} = \lim_{t\to\infty}\frac{D_t^2(x)}{t} + \lim_{t\to\infty}\frac{D_t^3(x)}{t}, \quad x>0, \qquad (3.52)$$

where $U_t^1(x)$ is the numbers of upcrossings of level x on page 1, $D_t^2(x)$ and $D_t^3(x)$ are the numbers of downcrossings of level x on pages 2 and 3 during the time interval $(0, t)$, respectively. By the LC argument, we have

$$\lim_{t\to\infty} \frac{U_t^1(x)}{t} = r_1 f_1(x), \quad \lim_{t\to\infty} \frac{D_t^2(x)}{t} = r_2 f_2(x), \quad \lim_{t\to\infty} \frac{D_t^3(x)}{t} = r_3 f_3(x). \tag{3.53}$$

Substituting (3.53) into (3.52) yields (3.49). Collecting the term $f_3(x)$ in (3.49) and substituting $f_3(x)$ into (3.40) gives us

$$f(x) = \left(1 + \frac{r_1}{r_3}\right) f_1(x) + \left(1 - \frac{r_2}{r_3}\right) f_2(x). \tag{3.54}$$

To express $f_1(x)$, $f_2(x)$, and $f_3(x)$ in terms of $f(x)$, we use the triangle diagram approach to find the long-run proportion of time for the activity periods. Figure 3.8 (a) and (b) demonstrates a sample path of the fluid level with leaking rate r_2 (with probability p_2) and r_3 (with probability p_3) given that the duration of activity period of the fluid queue is h_2 and h_3, respectively.

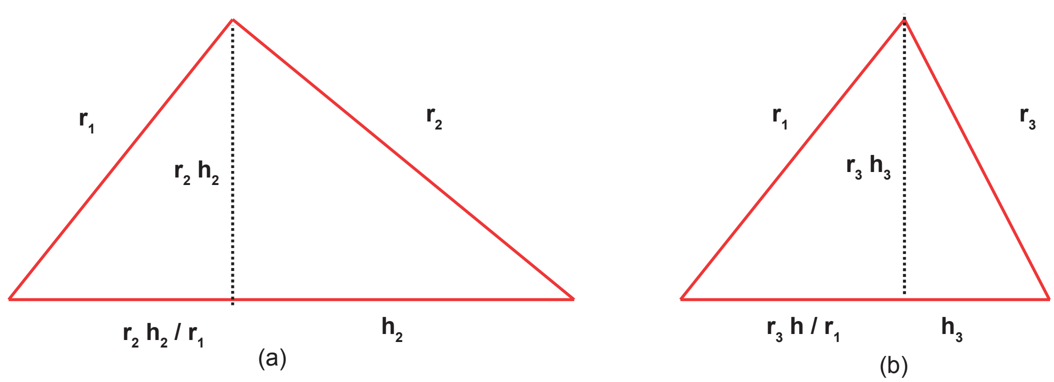

Figure 3.8: Triangle diagrams for a typical sample path

Observe in Figure 3.8 (a) and (b) that the proportion of time that the fluid level upcrosses and downcrosses level x are

$$\frac{r_2}{r_1 + r_2} \quad \text{and} \quad \frac{r_1}{r_1 + r_2}, \tag{3.55}$$

respectively. This suggests that

$$f_{1|2}(x) = \frac{r_2}{r_1 + r_2} f(x) \quad \text{and} \quad f_{2|2}(x) = \frac{r_1}{r_1 + r_2} f(x), \tag{3.56}$$

where $f(x)$ is the marginal density of the fluid level defined in (3.40), and $f_{i|j}(x)$ is the partial density defined as

$$f_{i|j}(x) = \frac{d}{dx} \lim_{t\to\infty} \mathbb{P}[\mathcal{C}(t) \leq x, Z(t) = i \mid \text{Leaking rate} = r_j], \quad i = 1, 2; j = 2, 3. \tag{3.57}$$

Using a similar approach, we have

$$f_{1|3}(x) = \frac{r_3}{r_1 + r_3} f(x) \quad \text{and} \quad f_{3|3}(x) = \frac{r_1}{r_1 + r_3} f(x). \tag{3.58}$$

For $\{\mathcal{C}(t) > 0\}_{t\geq 0}$, the leaking rate is independent of the fluid level and the state background processes $\{Z(t)\}_{t\geq 0}$ defined in (2.2). Thus by expressions (3.56) and (3.58), we have

$$f_1(x) = \frac{p_2 r_2}{r_1 + r_2} f(x) + \frac{p_3 r_3}{r_1 + r_3} f(x), \tag{3.59}$$

$$f_2(x) = \frac{p_2 r_1}{r_1 + r_2} f(x) \;\; \text{and} \;\; f_3(x) = \frac{p_3 r_1}{r_1 + r_3} f(x). \tag{3.60}$$

Substituting (3.60) into (3.46) gives us (for $x > 0$)

$$\begin{aligned} r_1(r_2 p_2 h_1 + r_3 p_3 h_2) f(x) &= \lambda \pi_{\mathcal{E}} h_1 h_2 \overline{B}\left(\frac{x}{r_1}\right) \\ &\quad + \lambda(p_2 h_1 + p_3 h_2) \int_{y=0}^{x} \overline{B}\left(\frac{x-y}{r_1}\right) f(y) dy, \end{aligned} \tag{3.61}$$

where $h_1 = r_1 + r_3$ and $h_2 = r_1 + r_2$.

3.2.3 Probability distribution of fluid level

For $s > 0$, define

$$\widetilde{\mathcal{C}}(s) = \pi_{\mathcal{E}} + \int_{0^+}^{\infty} e^{-sx} d\mathbb{P}[\mathcal{C} \leq x], \quad \text{and} \quad \widetilde{B}(s) = \int_{0}^{\infty} e^{-sx} d\mathbb{P}[B \leq x] \tag{3.62}$$

as the LST of the pdf of the fluid level and the pdf of the busy period of the driving queue, respectively. Multiplying e^{-sx} on both sides of (3.61) and integrating with respect to x over $(0, x)$ yields

$$\begin{aligned} r_1 s \left(r_2 p_2 h_1 + r_3 p_3 h_2\right) \left(\widetilde{C}(s) - \pi_{\mathcal{E}}\right) &= \lambda \pi_{\mathcal{E}} h_1 h_2 \left(1 - \widetilde{B}(r_1 s)\right) \\ &\quad + \lambda (p_2 h_1 + p_3 h_2) \left(1 - \widetilde{B}(r_1 s)\right) \left(\widetilde{C}(s) - \pi_{\mathcal{E}}\right). \end{aligned} \tag{3.63}$$

Collecting the term $\widetilde{C}(s)$ yields

$$\begin{aligned} \widetilde{C}(s) = \pi_{\mathcal{E}} \Bigg(& \frac{r_1 p_2 h_1 \left[r_2 s - \lambda \left(1 - \widetilde{B}(r_1 s)\right)\right] + r_1 p_3 h_2 \left[r_3 s - \lambda \left(1 - \widetilde{B}(r_1 s)\right)\right]}{r_1 \left[p_2 h_1 \left[r_2 s - \lambda \left(1 - \widetilde{B}(r_1 s)\right)\right] + p_3 h_2 \left[r_3 s - \lambda \left(1 - \widetilde{B}(r_1 s)\right)\right]\right]} \\ & + \frac{\lambda h_1 h_2 \left[1 - \widetilde{B}(r_1 s)\right]}{r_1 \left[p_2 h_1 \left[r_2 s - \lambda \left(1 - \widetilde{B}(r_1 s)\right)\right] + p_3 h_2 \left[r_3 s - \lambda \left(1 - \widetilde{B}(r_1 s)\right)\right]\right]} \Bigg). \end{aligned} \tag{3.64}$$

To find $\pi_{\mathcal{E}}$, first letting $s \to 0^+$ and applying L'Hôpital's Rule in (3.64) yields

$$\begin{aligned} \lim_{s \to 0^+} \widetilde{\mathcal{C}}(s) = \pi_{\mathcal{E}} \Bigg(& \frac{r_1 p_2 h_1 (r_2 - r_1 \lambda \mathbb{E}[B]) + r_1 p_3 h_2 (r_3 - r_1 \lambda \mathbb{E}[B])}{r_1 \left[p_2 h_1 (r_2 - r_1 \lambda \mathbb{E}[B]) + p_3 h_2 (r_3 - r_1 \lambda \mathbb{E}[B])\right]} \\ & + \frac{r_1 \lambda h_1 h_2 \mathbb{E}[B]}{r_1 \left[p_2 h_1 (r_2 - r_1 \lambda \mathbb{E}[B]) + p_3 h_2 (r_3 - r_1 \lambda \mathbb{E}[B])\right]} \Bigg). \end{aligned}$$

Then using a property of LST, namely $\widetilde{\mathcal{C}}(0^+) = 1$, gives us

$$\pi_{\mathcal{E}} = \frac{r_1 \left[p_2 h_1 (r_2 - r_1 \lambda \mathbb{E}[B]) + p_3 h_2 (r_3 - r_1 \lambda \mathbb{E}[B])\right]}{r_1 p_2 h_1 (r_2 - r_1 \lambda \mathbb{E}[B]) + r_1 p_3 h_2 (r_3 - r_1 \lambda \mathbb{E}[B]) + r_1 \lambda h_1 h_2 \mathbb{E}[B]}. \tag{3.65}$$

Remark 3.2.2. *Substituting $r_3 = r_2$ in (3.65) yields*

$$\pi_{\mathcal{E}} = \frac{r_2 - r_1\lambda\mathbb{E}[B]}{r_2 + r_2\lambda\mathbb{E}[B]}, \tag{3.66}$$

which equals the result in (2.34), namely the long-run proportion of time that the fluid level is at 0 *for the fluid queue defined in Section 2.1.1. The expression (3.66) is a necessary condition for (3.65) being correct.*

Figure 3.9 illustrates the LST of the pdf of the fluid level corresponding to five different sets of parameters $\mu = 3, \lambda = 2, p_2 = 0.3, p_3 = 1 - p_2, r_1 = 1, r_2 = 6$ with $r_3 = 6,\ 5.5,\ 5,\ 4.5,\ 4$. Without loss of generality, we assume that the driving queue for the fluid queue is an $M/M/1$ queue. It is important to note that when $r_3 = 6$, we have the ordinary fluid queue driven by an $M/M/1$ queue discussed in Section 2.1.1. As observed, the LST of the pdf of the fluid level for the ordinary fluid queue is the upper bond of the LST of the pdfs of the fluid level. In addition, as r_3 (< 6) decreases, the LST of the pdf of the fluid level decreases with respect to s, and $\pi_{\mathcal{E}}$ decreases as well. One explanation is that the long-run proportion of time of the wet period is proportion to the *continuous* leaking rate r_3, as the *continuous* leaking rate r_3 decreases, the long-run proportion of time of the wet period increases, thus $\pi_{\mathcal{E}}$ decreases. Using this argument, we can conclude that the LST of the pdf of the fluid level with $r_3 = 6$ is the upper bound of the LSTs of the pdf of the fluid level, since the sample path of the fluid level (with rate $r_3 = 6$) takes less time to reach level 0 than other sample path of the fluid levels (with $r_3 < 6$)., i.e. it has higher chance to finish the *game* before the catastrophe arrives.

Similarly, Figure 3.10 illustrates the LST of the pdf of the fluid level corresponding to five different sets of parameters $\mu = 3, \lambda = 2, p_2 = 0.3, p_3 = 1 - p_2, r_1 = 1, r_2 = 6$ with $r_3 = 6,\ 6.5,\ 7,\ 7.5,\ 8$. As observed, the LST of the pdf of the fluid level for the ordinary fluid queue is the lower bond of the LSTs of the pdf of the fluid level. In addition, as

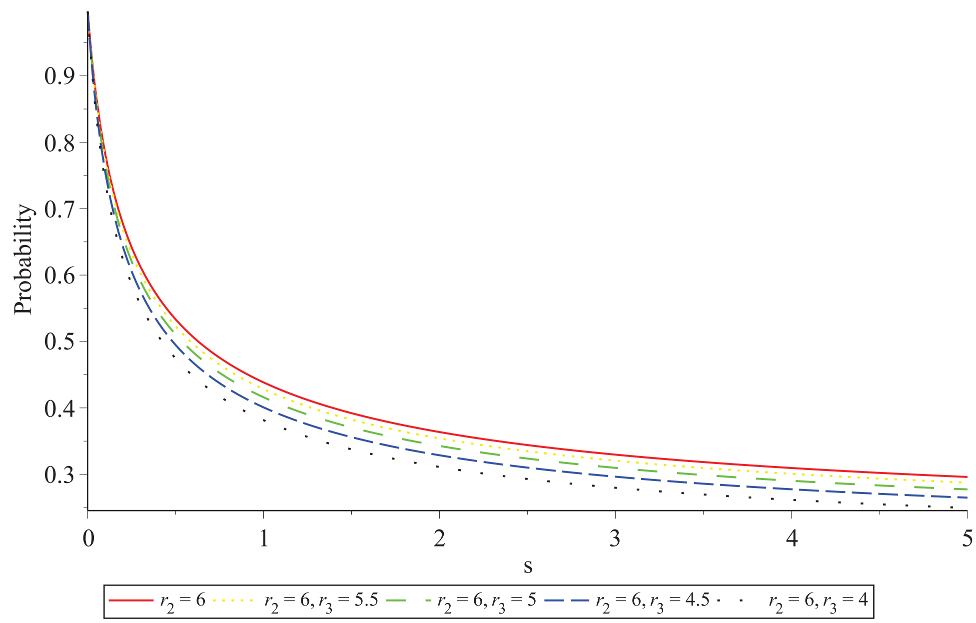

Figure 3.9: Laplace transform of fluid level

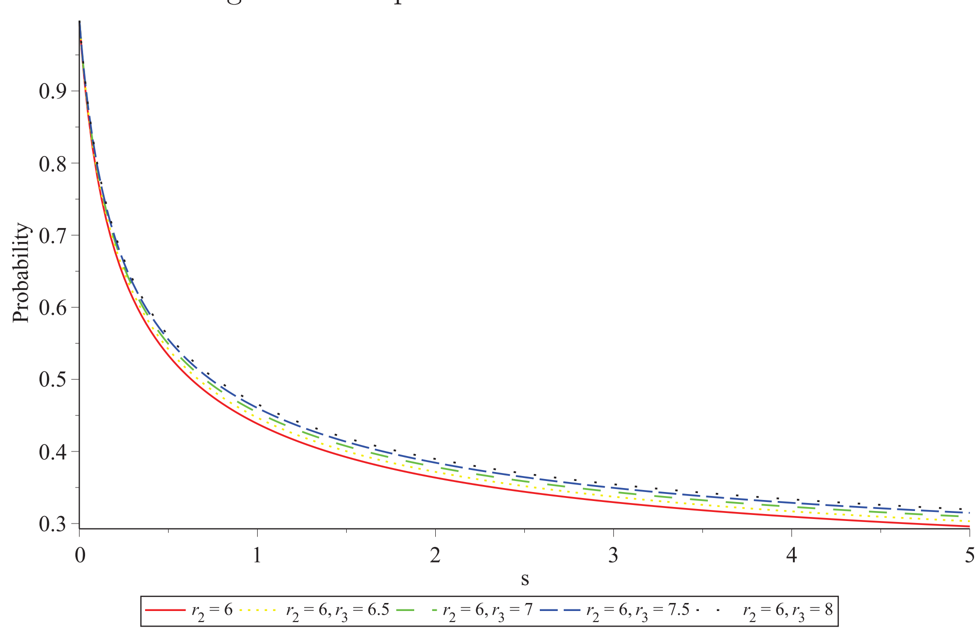

Figure 3.10: Laplace transform of fluid level

r_3 (> 6) increases, the LST of the pdf of the fluid level pdf increases with respected to s, and $\pi_{\mathcal{E}}$ increases as well. Finally, the LST of the pdf of the fluid level with rate $r_3 = 6$ is the lower bound of the other LSTs of the pdf of the fluid level can be explained using similar argument for the continuous leaking rate $r_3 < 6$.

Finally, Figures 3.11 and 3.12 illustrate the approximate pdf of fluid level for $x \in (0, 10]$ corresponding to Figures 3.9 and 3.10 respectively by using the EULER Method described in [1, 3].

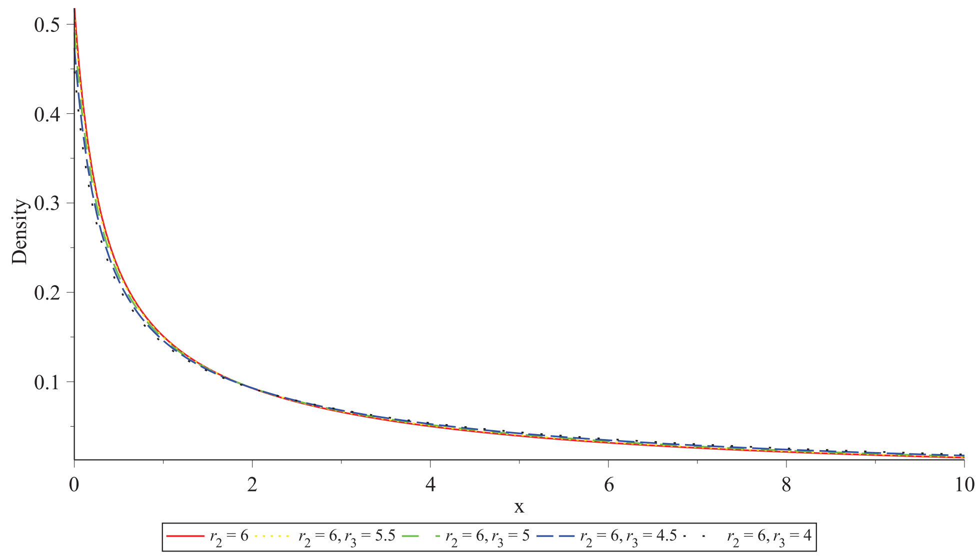

Figure 3.11: Approximated probability density functions corresponding to the LSTs in Figure 3.9

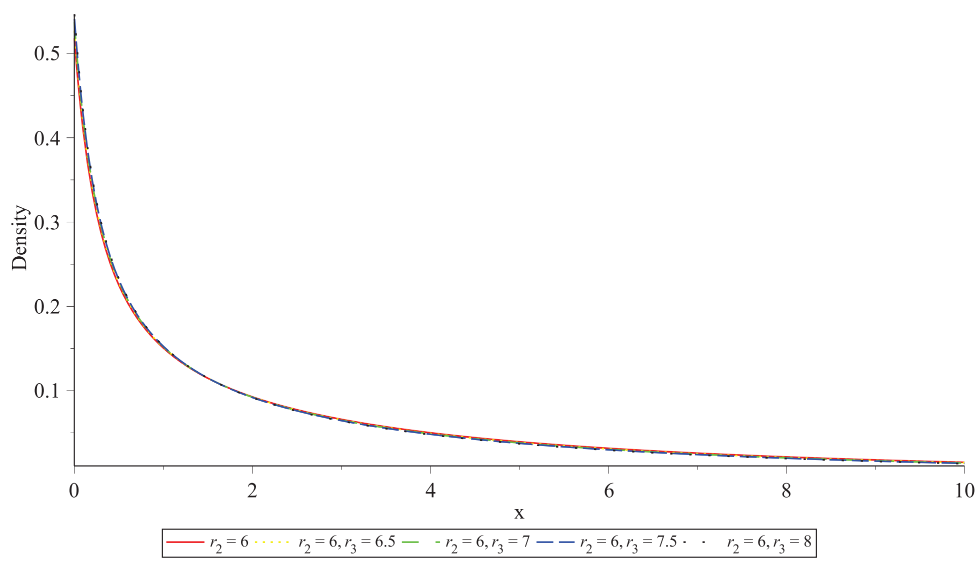

Figure 3.12: Approximated probability density functions corresponding to the LSTs in Figure 3.10

Taking derivatives with respect to s in (3.64) gives us

$$\widetilde{C}'(s) = \frac{\lambda\pi_{\mathcal{E}}(h_1+h_2)(p_2r_1h_1+p_3r_3h_2)\left(\widetilde{B}(r_1s)-r_1s\widetilde{B}'(r_1s)-1\right)}{r_1\left[p_2h_3\left(r_2s-\lambda\left(1-\widetilde{B}(r_1s)\right)\right)+p_3h_2\left(r_3s-\lambda\left(1-\widetilde{B}(r_1s)\right)\right)\right]^2}, \tag{3.67}$$

where $h_1 = r_1+r_3$ and $h_2 = r_1+r_2$. Multiplying ny -1 on both sides of above expression, and letting $s \to 0$ (and applying L'Hôpital's Rule two times) yields

$$-\widetilde{C}'(s)\Big|_{s=0} = \pi_{\mathcal{E}}\frac{r_1\lambda h_1h_3(p_2r_1r_2+p_2r_2r_3+p_3r_1r_3+p_3r_2r_3)\mathbb{E}[B^2]}{2\left(p_2h_1\left(r_2-\lambda r_1\mathbb{E}[B]\right)+p_3h_2\left(r_3-\lambda r_1\mathbb{E}[B]\right)\right)^2}, \tag{3.68}$$

which is the expected fluid level of this particular fluid queue.

Remark 3.2.3. *Letting $r_3 = r_2$ in (3.68) yields*

$$\mathbb{E}[\mathcal{C}] = \frac{r_1r_2(r_1+r_2)\lambda\mathbb{E}[B^2]}{2(r_2-\lambda\mathbb{E}[B])^2} \tag{3.69}$$

which equals the expression in (2.43) for the expected fluid level discussed in Section 2.3.2.

3.2.4 Expected value of fluid level

Denote by $\mathcal{W}$ and $\mathcal{E}$ the wet period and the idle period of the fluid queue. Using the *Renewal Reward Theorem* ([34, Prop. 7.3, p. 417]), often used in the LC method, the long-run proportion of time that $\{\mathcal{C}(t)\}_{t>0}$ is above level 0 is

$$\lambda\pi_{\mathcal{E}}\mathbb{E}[\mathcal{W}] = 1-\pi_{\mathcal{E}}, \tag{3.70}$$

and the long-run proportion of time that $\{\mathcal{C}(t)\}_{t>0}$ is at level 0 is

$$\lambda\pi_{\mathcal{E}}\mathbb{E}[\mathcal{E}] = \pi_{\mathcal{E}}. \tag{3.71}$$

Summing (3.70) and (3.71) gives

$$\lambda\pi_{\mathcal{E}}\mathbb{E}[\mathcal{W}_c] = 1, \tag{3.72}$$

where $\mathcal{W}_c$ is the wet cycle of the fluid queue. The expression (3.72) can be easily achieved by using the *Renewal Reward Theorem* ([34, Prop. 7.3, p. 417]).

Lemma 3.2.4. *Assume that fluid queue is stable. Then the expected wet period, idle period, and wet cycle of the fluid queue are determined by λ and $\pi_{\mathcal{E}}$, namely*

$$\mathbb{E}[\mathcal{W}] = \frac{1-\pi_{\mathcal{E}}}{\lambda\pi_{\mathcal{E}}},\ \mathbb{E}[\mathcal{E}] = \frac{1}{\lambda},\ \textit{and}\ \mathbb{E}[\mathcal{W}_c] = \frac{1}{\lambda\pi_{\mathcal{E}}}. \tag{3.73}$$

Using a simple LC argument, we express the expected value of wet cycle, wet period, and idle period in terms of λ and $\pi_{\mathcal{E}}$ without finding the distribution of the wet cycle and wet period[4].

3.2.5 Simulation of fluid level

In this section, we conduct a numerical study to illustrate the performance of the modified fluid queue and the ordinary fluid queue introduced in Section 2.1.1, in terms of their expected values ($\mathcal{C}$), wet periods ($\mathcal{W}$), and the point mass of fluid level at 0 ($\pi_{\mathcal{E}}$), using MAPLE 18. As usual, we assume that both fluid queues are driven by an $M/M/1$ queue with arrival rate $\lambda = 2.5$ and service rate $\mu = 6$. In addition, we assume that the *net* input rate $r_1 = 1$, the leaking rate $r_2 = 6$ with probability $p_2 = 0.3$, and the leaking rates r_3 with probability $p_3 = 1 - p_2$.

In terms of the simulation processes, each simulation contains $10,000$ arrivals in the

[4]Similar arguments can be used to find the busy cycle, busy period and idle period of an $M/G/1$ queue. See [12, Section 3.3.8, p. 71].

driving $M/M/1$ queue. For each complete wet cycle of the fluid queues, we compute the average fluid levels, wet and empty periods corresponding to the modified fluid queue. The empirical expected fluid level and wet period, and probability of fluid level being at level 0 are computed by aggregating the average fluid level for each simulated complete wet cycle, and duration of wet and empty periods in each simulated complete wet cycle. In terms of the theoretical expected values of fluid level, wet period and probability of being at level 0, we use (3.68), (3.73), and (3.65) respectively. The results of the simulation are reported in Table 3.1.

Table 3.1: Simulation results of fluid level

Parameters							Empirical results			Theoretical results		
λ	μ	r_1	r_2	r_3	p_2	p_3	$\mathbb{E}[\mathcal{C}]$	$\mathbb{E}[\mathcal{W}]$	$\pi_{\mathcal{E}}$	$\mathbb{E}[\mathcal{C}]$	$\mathbb{E}[\mathcal{W}]$	$\pi_{\mathcal{E}}$
2.5	6.0	1.0	6.0	4.0	0.3	0.7	0.311	0.435	0.478	0.310	0.434	0.479
2.5	6.0	1.0	6.0	4.5	0.3	0.7	0.297	0.415	0.490	0.296	0.415	0.490
2.5	6.0	1.0	6.0	5.0	0.3	0.7	0.286	0.400	0.499	0.285	0.400	0.500
2.5	6.0	1.0	6.0	5.5	0.3	0.7	0.278	0.388	0.507	0.277	0.388	0.507
2.5	6.0	1.0	6.0	6.0	0.3	0.7	0.271	0.378	0.513	0.270	0.378	0.513
2.5	6.0	1.0	6.0	6.5	0.3	0.7	0.265	0.370	0.518	0.264	0.370	0.519
2.5	6.0	1.0	6.0	7.0	0.3	0.7	0.260	0.363	0.523	0.259	0.363	0.523
2.5	6.0	1.0	6.0	7.5	0.3	0.7	0.256	0.358	0.527	0.255	0.357	0.527
2.5	6.0	1.0	6.0	8.0	0.3	0.7	0.252	0.353	0.531	0.252	0.352	0.531

As shown in Table 3.1, the expected fluid level and the expected wet period decrease as the leaking rate r_3 increases. Conversely, the probability of the fluid level being at level 0 increases as the leaking rate r_3 increases. It is important to note that for the modified fluid queue, when $r_2 = r_3 = 6$, it becomes the ordinary fluid queue discussed in Chapter 2. When $r_3 < 6$, the expected fluid level and wet period are larger than the expected fluid level and wet period of the ordinary fluid queue, whereas the expected fluid level and wet period are less than the expected fluid level and wet period of the ordinary fluid queue when $r_3 > 6$.

Chapter 4

Fluid queue with jump fluid inputs

In Chapters 2 and 3, we consider fluid queues where the fluid content fills at rate r_1 *continuously* when the server of the driving queue is busy, and the fluid content empties at rate r_2 *continuously* as long as the fluid content is non-empty. In this chapter, we consider two models of fluid queues in which the arrival process of the driving queue, in terms of accepting or rejecting a zero-wait arrival, is controlled by a fluid queue manager. If the system manager rejects the zero-wait arrivals, then the arrivals leave the driving queue without having a service. However, the fluid content is able to receive the fluid generated by these arrivals *instantly* when they leave the driving queue. For each model, the Laplace-Stieltje transforms (LST) of the probability density function (pdf) of the fluid level is derived for the fluid models.

4.1 Model I: Fluid queue with irregular arrivals

4.1.1 Motivation and introduction

Motivation. In Chapters 2 and 3, we study the characteristics of the fluid queue where the fluid enters and leaves the fluid content depending on the background process, and the fluid level. For these fluid queues, it is assumed that any arrival at the driving queue will receive a service, and the fluid content receives the fluid generated by these arrivals continuously with *net* input rate r_1 when the server of the driving queue is busy. In this chapter, we consider a fluid queue where the fluid content not only fills at rate r_1, but also the fluid content can receive the fluid instantly when an arrival in the driving queue is rejected by the system manager.

Introduction. In the classical fluid queue driven by a single server $M/G/1$ queue, let B denote the busy period of the driving $M/G/1$ queue. All busy periods initiated by the zero-wait arrivals in the driving queue contribute $r_1 B$ units of the fluid to the fluid content up to the end of the busy period in the driving queue. Suppose, however, that during an *empty* period of the fluid queue, the decision of accepting (with probability $1-p$) or rejecting (with probability p) a zero-wait arrival in the driving $M/G/1$ queue is made by a system manager. If the system manager decides to reject the zero-wait arrival in the driving queue, the rejected arrival will leave the driving queue without receiving a service. However, the fluid content in the fluid queue can still receive (*instantly*) the fluid generated by the rejected zero-wait arrival. Here, we assume that the magnitude of the fluid inputs generated by the rejected zero-wait arrivals equal the magnitude of their service times in the driving queue. We refer to the zero-wait arrivals who arrive at the time when the fluid content is empty and are rejected by the driving queue as *irregular zero-wait arrivals*, and those accepted by the system manager as *regular zero-wait arrivals.*

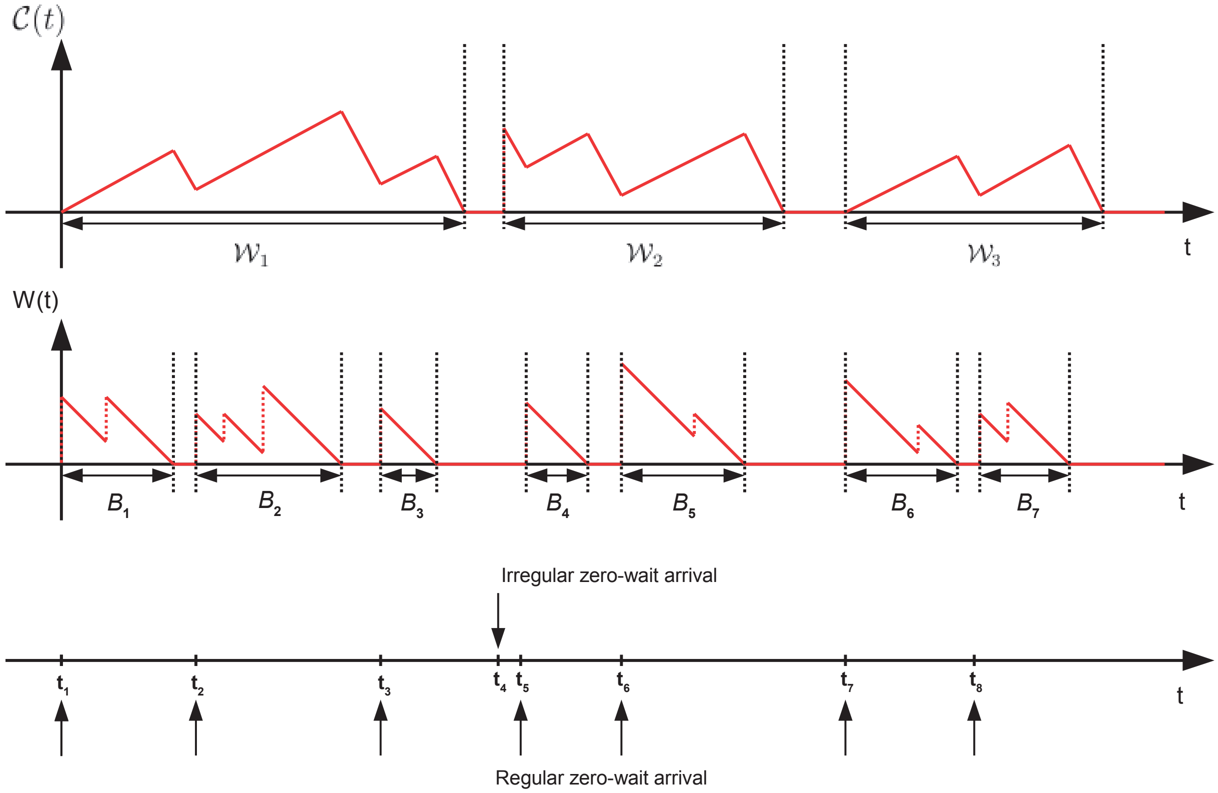

Figure 4.1: A sample path of fluid queue with irregular arrival

Figure 4.1 illustrates a sample path of the fluid level (top) and a sample path of the virtual waiting time of the driving queue (middle) corresponding to the fluid queue. The arrival times of the regular and irregular zero-wait arrivals are illustrated at the bottom of the Figure 4.1. Here, the fluid queue consists of 3 wet periods. The first wet period $\mathcal{W}_1$ encompasses three $M/G/1$ busy periods (B_1, B_2, and B_3 as indicated in Figure 4.1) without an irregular zero-wait arrival. Thus, the characteristics of this wet period is identical to the wet period of the ordinary fluid queue driven by an $M/G/1$ queue discussed in Section 2.1.1. The second wet period $\mathcal{W}_2$ encompasses one irregular zero-wait arrival followed by two $M/G/1$ busy periods. This irregular zero-wait arrival at time t_4 contributes his/her service time (here, we assume one unit of time is equal to one unit of fluid) to the fluid content without receiving a service in the driving queue. The characteristics of the wet period $\mathcal{W}_3$ are similar to $\mathcal{W}_1$, since no irregular zero-wait arrival is in $\mathcal{W}_3$. We refer to this particular fluid queue as a *fluid queue with irregular* *arrivals.*

4.1.2 Level crossing equations for $f_1(x), f_2(x)$, and $f(x)$

To derive the LST of the pdf of the fluid level for the fluid queue with irregular arrivals, we use LC methods [12]. Consider a single server $M/G/1$ queue with arrival rate λ and expected service time $1/\mu$. We assume that the services are independent and identically distributed (iid) with a common distribution G. During the *empty period* of the fluid queue, with probability p, the system manager rejects a zero-wait arrival. This rejected arrival causes a jump in the fluid content. To simplify our notation, we denote

$$\lambda_p = p \cdot \lambda \quad \text{and} \quad \lambda_q = q \cdot \lambda, \tag{4.1}$$

where $q = 1 - p$. We remark that these two arrival rates apply only to the zero-wait arrivals who arrive during the empty periods of the fluid queue.

Crossing rates of the fluid level x. Denote by $U_t^c(x)$ the number of upcrossings (*continuously*) of level x, by $U_t^j(x)$ the number of upward jumps (*instantly*) of level x, and by $D_t(x)$ the number of downcrossings (*continuously*) of level x, during the time interval $(0, t)$. For a fixed fluid level $x > 0$, by a LC argument [12, p. 109, and pp. 304 - 309], the rates at which the sample path upcrosses and downcrosses level x during the time interval $(0, t)$ are

$$\lim_{t\to\infty} \frac{U_t^c(x)}{t} = r_1 f_1(x) \quad \text{plus} \quad \lim_{t\to\infty} \frac{U_t^j(x)}{t} = \lambda_p \pi_{\mathcal{E}} \overline{G}(x), \tag{4.2}$$

$$\lim_{t\to\infty} \frac{D_t(x)}{t} = r_2 f_2(x), \tag{4.3}$$

where $f_1(x)$ and $f_2(x)$ are defined in (2.5) respectively, $\overline{G}(x)$ is the complementary cumulative distribution function (ccdf) of the service in the driving queue (i.e., the magnitude of the jump in the fluid queue), and $\pi_{\mathcal{E}}$ is the point mass for the fluid level at 0. The terms $\lim_{t\to\infty} U_t^c(x)/t$ in (4.2) and $\lim_{t\to\infty} D_t(x)/t$ in (4.3) can be interpreted in a similar manner as in (2.9). The term $\lim_{t\to\infty} U_t^j(x)/t$ in (4.2) can be interpreted as follows: the rate at which a zero-wait arrival is rejected by the fluid queue manager, i.e. the rate at which an irregular zero-wait arrival causes a jump above level x in the fluid content. It is important to highlight that the jump in the fluid content can only start at level 0. Let $\lim_{t\to\infty} U_t(x)/t$ denote the total upcrossing (continuous and jump) rate of level x, then

$$\lim_{t\to\infty} \frac{U_t(x)}{t} = \lim_{t\to\infty} \frac{U_t^c(x)}{t} + \lim_{t\to\infty} \frac{U_t^j(x)}{t} = r_1 f_1(x) + \lambda_p \pi_{\mathcal{E}} \overline{G}(x), \quad x > 0. \tag{4.4}$$

Downcrossing rate of level x toward level zero. As indicated in (4.2), there are two different ways for the sample path to leave level 0: (1) upward jump at level 0 (i.e. upward jumps caused by an *irregular zero-wait arrival*), and (2) sloped upcrossing of level 0 (i.e. an activity period is initiated by a regular *zero-wait arrival*). However, there is only one way to downcross level x, namely with slope r_2. Balancing the downcrossing rate of level x and the total upcrossing rate of level x , we obtain

$$r_2 f_2(x) = r_1 f_1(x) + \lambda_p \pi_{\mathcal{E}} \overline{G}(x), \quad x > 0. \tag{4.5}$$

It is important to highlight that the jumps caused by the irregular zero-wait arrivals are strictly positive. Thus, we have $\overline{G}(0^+) = 1$. Letting $x \to 0^+$ in (4.5) yields

$$(\lambda_p + \lambda_q)\pi_{\mathcal{E}} = r_1 f_1(0^+) + \lambda_p \pi_{\mathcal{E}}, \tag{4.6}$$

with $\lambda_p + \lambda_q = \lambda$.

In addition, since the rate at which the sample path leaves level 0 equals the rate at which the sample path enters level 0, we have

$$(\lambda_p + \lambda_q)\pi_{\mathcal{E}} = r_2 f_2(0^+). \tag{4.7}$$

The term $\lambda\pi_{\mathcal{E}} = (\lambda_p + \lambda_q)\pi_{\mathcal{E}}$ is the rate at which sample path leaves level 0. Let *gram* be the unit of the fluid queue and T be the unit of the time; a necessary condition on equality (4.6) and (4.7) is that the units of these two expressions are the same, i.e. $[1/T]$. Substituting the units for the rates and the units for the fluid into (4.6) and (4.7), the unit of (4.6) and (4.7) is

$$\left[\frac{gram}{T} \cdot \frac{1}{gram}\right] = \left[\frac{1}{T}\right] \quad \text{and} \quad \left[\frac{gram}{T} \cdot \frac{1}{gram}\right] + \left[\frac{1}{T}\right] = \left[\frac{1}{T}\right] \tag{4.8}$$

respectively. The interpretation of expression (4.6) is equating two different expressions of the rate at which the sample path leaves level 0. Simplifying (4.6) yields

$$\lambda_q \pi_{\mathcal{E}} = r_1 f_1(0^+). \tag{4.9}$$

The term on the left-hand side of (4.9) is the rate at which the sample path upcrosses (continuously) level x starting from level 0. It implies that $r_1 f_1(0^+)$ is the rate at which the sample path (continuous) leaves level 0 (but not the rate at which the sample path leaves level 0 by an upward jump).

Denote by $\mathcal{C}(t)$ the fluid level at time t. Each instant when $\{\mathcal{C}(t)\}_{t\geq 0}$ *enters* level 0 is a regenerative point, due to Poisson arrivals in the driving process, and the memoryless property of the exponential distribution. Thus $\{\mathcal{C}(t)\}_{t\geq 0}$ is a regenerative process with wet cycles $\{\mathcal{W}_c\}$ that form renewal process. Denote by $\mathcal{W}$ and $\mathcal{W}_c$ the wet period and the wet cycle of the fluid queue. By the *Elementary Renewal Theorem* [34, Section 7.3, pp. 407 - 416], $\mathbb{E}[\mathcal{W}_c] = 1/(\lambda\pi_{\mathcal{E}})$. By the *Renewal Reward Theorem* [34, Proposition 7.3, pp. 416 - 417], the long-run proportion of time that the fluid level is at level 0 is

$\mathbb{E}[\mathcal{E}]/(\mathbb{E}[\mathcal{E}]+\mathbb{E}[\mathcal{W}])$, where $\mathcal{E}$ is the empty period of the fluid queue with expected value $\mathbb{E}[\mathcal{E}]=1/\lambda$. It follows that once we get $\pi_{\mathcal{E}}$, we get the $\mathbb{E}[\mathcal{W}]$ and $\mathbb{E}[\mathcal{W}_c]$.

Upcrossing rate of level x in the fluid queue: Let $f(x)=f_1(x)+f_2(x)$ denote the total pdf of the fluid level, where $f_1(x)$ and $f_2(x)$ are defined in (2.5). We refer to the sample path corresponding to $f_1(x)$ as a sample path on 'page 1', and to the sample path corresponding to the $f_2(x)$ as a sample path on 'page 2'. For the details of the method of page *pages*, the reader is referred to Brill [12, Chapter 4, pp. 162 - 254, and Section 10.10, pp. 433 - 439]. Using a LC argument, the total rate at which the sample path upcrosses level x can be written in *two different expressions*, namely:

$$\lim_{t\to\infty}\frac{U_t(x)}{t}=r_1 f_1(x), \tag{4.10}$$

$$\lim_{t\to\infty}\frac{U_t(x)}{t}=\lambda_p\pi_{\mathcal{E}}\overline{G}(x)+\lambda_q\pi_{\mathcal{E}}\overline{B}\left(\frac{x}{r_1}\right)+\lambda\int_{y=0}^{x}\overline{B}\left(\frac{x-y}{r_1}\right)f_2(y)dy. \tag{4.11}$$

The right-hand side of (4.10) is the rate at which the sample path upcrosses level x on page 1 (see [12, p. 304, eq (6.5) and Section 10.10.2, p. 435] for details). The first term on the right-hand side of (4.11) is the rate at which the sample path *jumps* across level x from level 0. The second term on the right-hand side of (4.11) is the rate at which the sample path upcrosses level x at slope r_1 starting from level 0, and the third term on the right-hand side of (4.11) is the rate at which the sample path upcrosses level x at slope r_1 starting from level y in $(0,x)$ on page 1 due to an instantaneously parallel jump from page 2. The terms $(x-y)/r_1$ in $\overline{B}(\bullet)$ is the time needed to upcross level x starting at level $y\in(0,x)$.

Downcrossing rate of level x in the fluid queue: The rate at which the sample path downcrosses level x is

$$\lim_{t\to\infty} \frac{D_t(x)}{t} = r_2 f_2(x). \tag{4.12}$$

The term on the right-hand side of (4.12) is the rate at which the sample path downcrosses level x on page 2. All downcrossings of level x occur on page 2.

4.1.3 Laplace-Stieltjes transform of fluid level

Solution for $f_2(x)$: Let $\mathcal{L}_{f_2}(s)$ denote the LST of the pdf $f_2(x)$. Setting (4.11) equal to (4.12), and taking the LST of the both sides of (4.11) and (4.12) yields

$$\begin{aligned} r_2(\mathcal{L}_{f_2}(s) - \pi_{\mathcal{E}}) = \lambda_p \pi_{\mathcal{E}} \cdot \frac{1-\widetilde{G}(s)}{s} &+ \lambda_q \pi_{\mathcal{E}} \cdot \frac{1-\widetilde{B}(r_1 s)}{s} \\ &+ \lambda \cdot \frac{1-\widetilde{B}(r_1 s)}{s} \cdot (\mathcal{L}_{f_2}(s) - \pi_{\mathcal{E}}). \end{aligned} \tag{4.13}$$

It is important to highlight that the unit of the parameter s in $\widetilde{G}(s)$ and $\widetilde{B}(r_1 s)$ are $[1/gram]$ instead of $[1/T]$, as expected. This is because the unit of the fluid level is $[gram]$, but not *time*. More specifically, the term $\widetilde{G}(s)$ is the LST of the pdf of the jump fluid input, thus the unit of s is $[1/gram]$. Yet, the term $\widetilde{B}(r_1 s)$ is the LST of the pdf of the busy period, thus the unit of $r_1 s$ is $[1/T]$. This gives the unit of s as $[1/gram]$, since the unit of r_1 is $[gram/T]$. Keeping this in mind, the unit of both sides of (4.13) is

$$\left[\frac{gram}{T}\right] = \left[\frac{gram}{T}\right] + \left[\frac{gram}{T}\right] + \left[\frac{gram}{T}\right] = \left[\frac{gram}{T}\right]. \tag{4.14}$$

This satisfies a necessary condition for the LC equation, and thus increases the confidence that the expression (4.13) is valid. Collecting the terms $\mathcal{L}_{f_2}(s)$ yields

$$\mathcal{L}_{f_2}(s) = \pi_{\mathcal{E}} \frac{sr_2 + \lambda_p(1 - \widetilde{G}(s)) + (\lambda_q - \lambda)(1 - \widetilde{B}(r_1 s))}{sr_2 - \lambda(1 - \widetilde{B}(r_1 s))}, \quad s > 0. \tag{4.15}$$

To see that the expression (4.15) is reasonable, we check whether the expression on the right-hand side of (4.15) is dimensionless. Evaluating the unit of the expression (4.15), we have

$$\left[\frac{1/T + 1/T + 1/T}{1/T - 1/T}\right] = [1], \tag{4.16}$$

which is dimensionless.

Solution for $f(x)$: Substituting $f(x) = f_1(x) + f_2(x)$ into (4.5), we obtain

$$r_1 f(x) = (r_1 + r_2) f_2(x) - \lambda_p \pi_{\mathcal{E}} \overline{G}(x). \tag{4.17}$$

Let $\widetilde{C}(s) = \pi_{\mathcal{E}} + \int_{0^-}^{\infty} e^{-sx} f(x) dx$ be the LST of the pdf of the fluid level. Taking LST of both sides of (4.17) yields

$$sr_1(\widetilde{C}(s) - \pi_{\mathcal{E}}) = s(r_1 + r_2)(\mathcal{L}_{f_2}(s) - \pi_{\mathcal{E}}) - \lambda \pi_{\mathcal{E}}(1 - \widetilde{G}(s)). \tag{4.18}$$

Simplifying the above expression yields

$$\widetilde{C}(s) = \frac{(r_1 + r_2)(\mathcal{L}_{f_2}(s) - \pi_{\mathcal{E}})}{r_1} - \frac{\lambda \pi_{\mathcal{E}}(1 - \widetilde{G}(s))}{sr_1} + \pi_{\mathcal{E}}, \tag{4.19}$$

which is the LST of the pdf of the fluid level. Recalling that the unit of sr_1 is $[1/T]$ gives us that the right-hand side of (4.19) is dimensionless. Substituting (4.15) into (4.19), we obtain

$$\widetilde{C}(s) = \left(\frac{r_1+r_2}{r_1}\right)\left(\pi_{\mathcal{E}}\frac{\lambda_p(1-\widetilde{G}(s))+\lambda_q(1-\widetilde{B}(r_1 s))}{sr_2-\lambda(1-\widetilde{B}(r_1 s))}\right) - \frac{\lambda\pi_{\mathcal{E}}(1-\widetilde{G}(s))}{sr_1} + \pi_{\mathcal{E}} \tag{4.20}$$

Applying L'Hôpital's rule in (4.20) yields

$$\frac{1}{\pi_{\mathcal{E}}}\lim_{s\to 0}\widetilde{C}(s) = \lim_{s\to 0}\left(\left(\frac{r_1+r_2}{r_1}\right)\left(\frac{-\lambda_p\widetilde{G}'(s)-\lambda_q r_1\widetilde{B}'(s)}{r_2+\lambda r_1\widetilde{B}'(s)}\right) + \frac{\lambda\widetilde{G}'(s)}{r_1} + 1\right), \tag{4.21}$$

which gives

$$\pi_{\mathcal{E}} = \left(\frac{(r_1+r_2)(\lambda_p\mathbb{E}[G]+\lambda_q r_1\mathbb{E}[B])}{r_1 r_2 - \lambda r_1^2\mathbb{E}[B]} - \frac{\mathbb{E}[G]}{r_1} + 1\right)^{-1}. \tag{4.22}$$

To verify that the expression (4.22) is reasonable, we let $p \to 0$ such that $\lambda_p \to 0$ and $\lambda_q \to \lambda$. The above expression becomes the point mass at level 0 for the ordinary fluid queue driven by an $M/G/1$ queue in (2.34). It is interesting to check the unit of the right-hand side of (4.22). This gives us[1]

$$\left[\frac{\frac{gram}{T}\left(\frac{1}{T}\cdot gram + \frac{1}{T}\cdot\frac{gram}{T}\cdot T\right)}{\left(\frac{gram}{T}\right)^2 - \frac{1}{T}\cdot\left(\frac{gram}{T}\right)^2\cdot T}\right]^{-1} = [1], \tag{4.23}$$

which is a correct dimension for $\pi_{\mathcal{E}}$.

[1]The dimension of $\mathbb{E}[G]$ is *gram*, not *time*, and the dimension of $\mathbb{E}[B]$ is *time*, not *gram*.

4.1.4 Mapping the sample path of the virtual waiting time of an $M/G/1$ with exceptional service for zero-wait arrivals to the sample path of the fluid level of the fluid queue with irregular arrivals

In this section, we determine the pdf of the virtual waiting time of an $M/G/1$ queue with exceptional service for zero-wait arrivals by using the pdf of the fluid level with irregular zero-wait arrivals. Let S_0 and S_1 denote the service time for the zero-wait arrivals and non-zero wait arrivals of an $M/G/1$ queue respectively. Figure 4.2 illustrates the virtual waiting time of the $M/G/1$ queue with exceptional service for zero-wait arrivals.

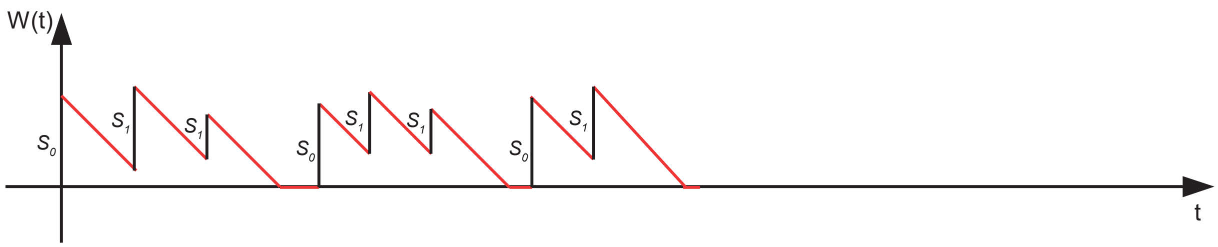

Figure 4.2: A sample path of virtual waiting time of an $M/G/1$ queue with exceptional service for zero-wait arrivals

To map the integral equation of the pdf of the virtual waiting time of the $M/G/1$ queue with exceptional service for zero-wait arrivals to the integral equation of the pdf of the fluid level of the fluid queue with irregular arrivals (Model I), we first set the *net* input rate r_1 and the *continuous* leaking rate r_2 of the fluid queue equal to 1.

A sample path of the fluid level of the fluid queue is illustrated at Figure 4.3. As observed in the figure, since the *net* input rate is 1, the magnitude of the *net* fluid that enters to the fluid content during the activity period equals the magnitude of the duration of the busy period of the driving $M/G/1$ queue. In addition, one zero-wait arrival in the driving queue is rejected by the system manager at time t_k with probability p. Balancing

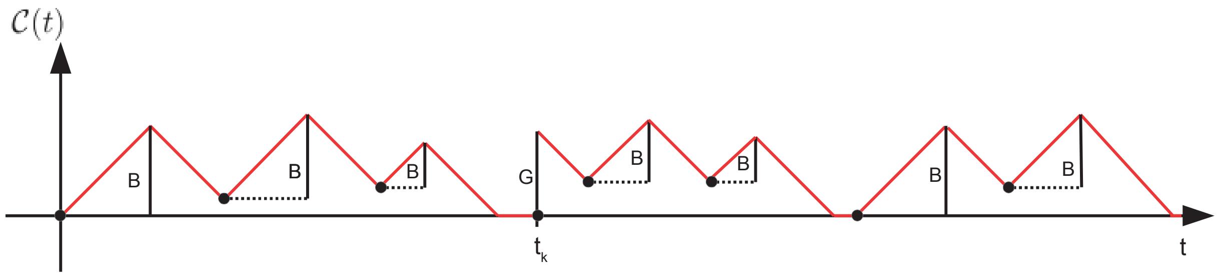

Figure 4.3: A sample path of virtual waiting time of an $M/G/1$ queue with exceptional service for zero-wait arrivals

the downcrossing rate (4.12) and upcrossing rate (4.11) of level x of the sample path of the fluid level, we have

$$f_2(x) = \lambda_p \pi_{\mathcal{E}} \overline{G}(x) + \lambda_q \pi_{\mathcal{E}} \overline{B}(x) + \lambda \int_{y=0}^{x} \overline{B}(x-y) f_2(y) dy, \quad x > 0. \tag{4.24}$$

Next, we fix $p = 1$ (i.e., $\lambda_p = \lambda$ and $\lambda_q = 0$) such that all zero-wait arrivals in the driving queue who initiate the activity period of the wet cycle are rejected. In addition, we assume that the size of the fluid jumps caused by the irregular arrivals are distributed as S_0 (such that $\overline{G}(x) = \overline{S_0}(x)$), and the busy period of the driving queue B is distributed as S_1 (such that $\overline{B}(x) = \overline{S_1}(x)$). The sample path of the fluid level of the fluid queue after these adjustments is illustrated in Figure 4.4. As observed in the figure, each time point when arrivals rejected by the system manger of the fluid queue is a regenerative points of the fluid process $\{\mathcal{C}(t)\}_{t\geq 0}$. Substituting $\lambda_p = p, \lambda_q = 0, \overline{B}(s) = \overline{S_1}(x)$, and

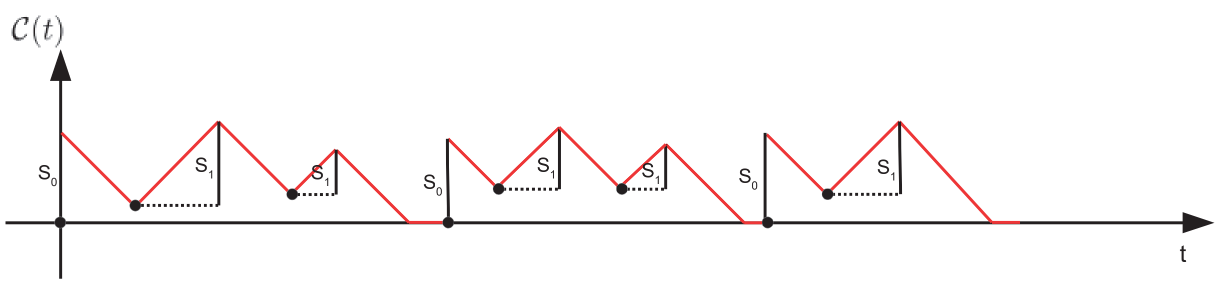

Figure 4.4: A sample path of fluid level of the fluid queue

$\overline{G}(x) = \overline{S}_0(x)$ into (4.24) gives

$$f_2(x) = \lambda \pi_{\mathcal{E}} \overline{S_0}(x) + \lambda \int_{y=0}^{x} \overline{S_1}(x-y) f_2(y) dy, \quad x > 0. \tag{4.25}$$

The integral equation of the pdf of the virtual waiting time of the $M/G/1$ queue with exceptional service for zero-wait arrivals is well-known (see [12, p. 99, eq (3.118)] for the details), namely

$$g(x) = \lambda \pi_0 \overline{S_0}(x) + \lambda \int_{y=0}^{x} \overline{S_1}(x-y) g(y) dy, \quad x > 0, \tag{4.26}$$

where π_0 is the point mass of the virtual waiting time of the queue at level 0..

As observed, the form of the integral equation of the pdf of the virtual waiting time in (4.26) has the same form of the integral equation of the pdf of the fluid level on 'page' 2 as expressed in (4.25) with one exception. The term $\pi_{\mathcal{E}}$ in (4.25) is the point mass of fluid level at level 0. Yet, the term π_0 in (4.26) is the long-run proportion of time that the virtual waiting time of the $M/G/1$ queue is at level 0.

4.2 Model II: Fluid queue with variant

To illustrate the flexibility of the LC method, we modify the fluid queue model in Section 4.1 with jump fluid inputs. Assume the arrival process of the driving queue is controlled by the fluid queue and that the system manager can accept (with probability $1-p$) and reject (with probability p) ***any*** zero-wait arrival at the driving queue when the fluid level of the fluid queue is at any level. One characteristic of the fluid queue is that the fluid content can receive the fluid (instantly) generated by a rejected zero-wait arrival. The significant difference between this fluid queue model and the one introduced in Section 4.1 is that in this particular fluid queue, the system manager can accept or reject zero-wait arrivals in the driving queue at any fluid level.

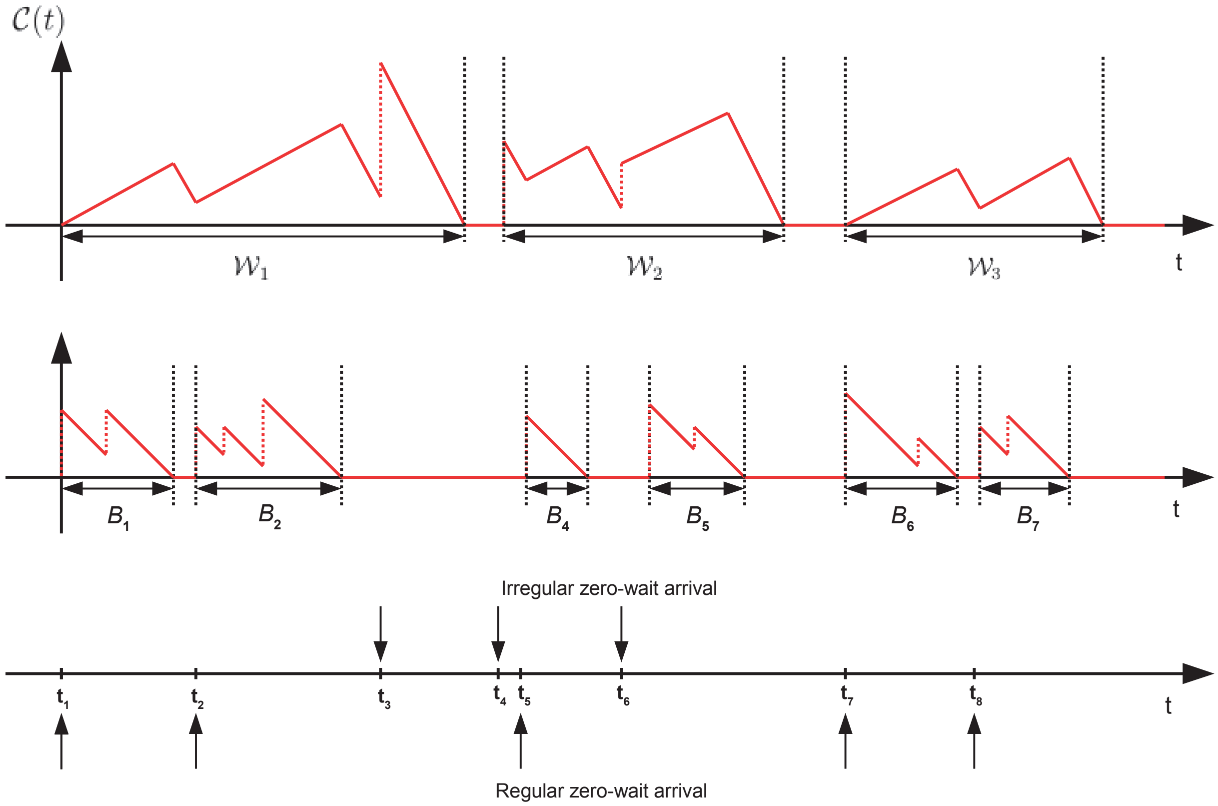

Figure 4.5: A sample path of fluid queue with irregular arrival

Figure 4.5 illustrates a sample path of the fluid level and a sample path of the virtual waiting time of the driving queue corresponding to the fluid queue. Each wet cycle in Figure 4.5 encompasses two busy periods in the driving queue. Here, as observed, one irregular arrival at time t_3 in the first wet period causes an upward jump. For the second wet period, two irregular arrivals at time t_4 and t_6 cause upward jumps in the fluid queue. There are no irregular arrivals in the third wet period.

Rate balance equation: Balancing the downcrossing rate and total upcrossing rate of level x, we obtain

$$r_2 f_2(x) = r_1 f_1(x) + \lambda_p \pi_{\mathcal{E}} \overline{G}(x) + \lambda_p \int_{y=0^+}^{x} \overline{G}(x-y) f_2(y) dy. \tag{4.27}$$

The first term on the right-hand side of (4.27) is the upcrossing (continuously) rate of level x; the second and third term on the right-hand side of (4.27) are the rates of the upward jumps above level x starting at level 0 and at some point $y \in (0, x)$ when the driving queue encounters zero-wait arrivals. To express $f_2(x)$ in terms of $f(x)$ in (4.27), we substitute $f_1(x) = f(x) - f_2(x)$ into (4.27). This yields

$$r_1 f(x) = (r_1 + r_2) f_2(x) - \lambda_p \pi_{\mathcal{E}} \overline{G}(x) - \lambda_p \int_{y=0^+}^{x} \overline{G}(x-y) f_2(y) dy. \tag{4.28}$$

Let $\widetilde{C}(s)$ be the LST of the pdf of the fluid level. Taking the LST on both sides of (4.28) and collecting the terms yields

$$\widetilde{C}(s) = \frac{s r_1 \pi_{\mathcal{E}}}{s r_1 + \lambda_p (1 - \widetilde{G}(s))} + \frac{(r_1 + r_2)(\mathcal{L}_{f_2}(s) - \pi_{\mathcal{E}})}{s r_1 + \lambda_p (1 - \widetilde{G}(s))}, \quad s > 0, \tag{4.29}$$

where $\pi_{\mathcal{E}}$ is the point mass of the fluid at level 0.

Alternatively, the rate balance equation in (4.27) can be written differently, namely:

$$\begin{aligned} r_2 f_2(x) = {} & \lambda_p \pi_{\mathcal{E}} \overline{G}(x) + \lambda_p \int_{y=0}^{x} \overline{G}(x-y) f_2(y) dy \\ & + \lambda_q \pi_{\mathcal{E}} \overline{B}\left(\frac{x}{r_1}\right) + \lambda_q \int_{y=0}^{x} \overline{B}\left(\frac{x-y}{r_1}\right) f_2(y) dy, \quad x > 0. \end{aligned} \tag{4.30}$$

The first two terms on the right-hand side of (4.30) are the rates at which the sample path jumps across level x, and the last two terms on the right-hand side of (4.30) are the rates at which the sample path upcrosses (*continuously*) level x.

LST of the pdf of the fluid level: Taking LST of both sides of (4.30) yields

$$\begin{aligned} s r_2 (\mathcal{L}_{f_2}(s) - \pi_{\mathcal{E}}) = {} & \lambda_p \pi_{\mathcal{E}} (1 - \widetilde{G}(s)) + \lambda_p (1 - \widetilde{G}(s))(\mathcal{L}_{f_2}(s) - \pi_{\mathcal{E}}) \\ & + \lambda_q \pi_{\mathcal{E}} (1 - \widetilde{B}(r_1 s)) + \lambda_q (1 - \widetilde{B}(r_1 s))(\mathcal{L}_{f_2}(s) - \pi_{\mathcal{E}}). \end{aligned} \tag{4.31}$$

Evaluating the unit of the expression (4.31) yields

$$\left[\frac{1}{gram}\cdot\frac{gram}{T}\right] = \left[\frac{1}{T}\right] + \left[\frac{1}{T}\right] + \left[\frac{1}{T}\right] + \left[\frac{1}{T}\right] = \left[\frac{1}{T}\right], \tag{4.32}$$

which is a necessary condition for the expression (4.31) to be valid. Collecting the terms $\mathcal{L}_{f_2}(s)$ yields

$$\mathcal{L}_{f_2}(s) = \frac{sr_2\pi_{\mathcal{E}}}{sr_2 - \lambda_p(1-\widetilde{G}(s)) - \lambda_q(1-\widetilde{B}(r_1 s))}, \quad s > 0. \tag{4.33}$$

Substituting (4.33) into (4.27) yields (for $s > 0$)

$$\begin{aligned}\widetilde{C}(s) = &\frac{sr_1\pi_{\mathcal{E}}}{sr_1 + \lambda_p(1-\widetilde{G}(s))} \\ &+ \left(\frac{(r_1+r_2)\pi_{\mathcal{E}}}{sr_1 + \lambda_p(1-\widetilde{G}(s))}\right)\left(\frac{\lambda_p(1-\widetilde{G}(s)) + \lambda_q(1-\widetilde{B}(r_1 s))}{sr_2 - \lambda_p(1-\widetilde{G}(s)) - \lambda_q(1-\widetilde{B}(r_1 s))}\right).\end{aligned} \tag{4.34}$$

Applying L'Hôpital's rule in (4.34) yields

$$\begin{aligned}\frac{1}{\pi_{\mathcal{E}}}\lim_{s\to 0}\widetilde{C}(s) &= \frac{r_1}{r_1+\lambda_p\mathbb{E}[G]} + \frac{r_1+r_2}{r_1+\lambda_p\mathbb{E}[G]}\cdot\left(\frac{\lambda_p\mathbb{E}[G]+\lambda_q r_1\mathbb{E}[B]}{r_2-\lambda_p\mathbb{E}[G]-\lambda_q r_1\mathbb{E}[B]}\right) \\ &= \frac{r_2(r_1+\lambda_p\mathbb{E}[G]+\lambda_q r_1\mathbb{E}[B])}{(r_1+\lambda_p\mathbb{E}[G])(r_2-\lambda_p\mathbb{E}[G]-\lambda_q r_1\mathbb{E}[B])}.\end{aligned} \tag{4.35}$$

This gives

$$\pi_{\mathcal{E}} = \frac{(r_1+\lambda_p\mathbb{E}[G])(r_2-\lambda_p\mathbb{E}[G]-\lambda_q r_1\mathbb{E}[B])}{r_2(r_1+\lambda_p\mathbb{E}[G]+\lambda_q r_1\mathbb{E}[B])}. \tag{4.36}$$

Here, the unit of the right-hand side of (4.36) is

$$\left[\frac{\left(\frac{gram}{T}+\frac{1}{T}\cdot gram\right)\cdot\left(\frac{gram}{T}-\frac{1}{T}\cdot gram-\frac{1}{T}\cdot\frac{gram}{T}\cdot T\right)}{\frac{gram}{T}\cdot\left(\frac{gram}{T}+\frac{1}{T}\cdot gram+\frac{1}{T}\cdot\frac{gram}{T}\cdot T\right)}\right]=[1], \tag{4.37}$$

which is the correct dimension for $\pi_{\mathcal{E}}$.

Letting $p \to 0$ such that $\lambda_p \to 0$ and $\lambda_q \to \lambda$, we have

$$\pi_{\mathcal{E}} = \frac{r_2 - r_1\lambda\mathbb{E}[B]}{r_2 + r_2\lambda\mathbb{E}[B]} \tag{4.38}$$

which is the point mass of the fluid level at 0 in (2.34).

Chapter 5

Summary of chapters

Here, we briefly summarize each chapter.

5.1 Summary of Chapter 1

In this dissertation, we study the characteristics of fluid queues driven by a stochastic background processes (such as $M/G/1$ queue or $M/G/1/1$ queue), and map the characteristics of the $M/G/1$ queue to the fluid queue using the *Level Crossing* method proposed by Brill in 1974 (see [12]). Level crossing methods allow researchers to write down an integral equation of the probability density function (pdf) of a process by studying the characteristics of the sample path corresponding to the process that is under investigation. Chapter 1 provides a basic review of renewal processes and the level crossing methods. Applications of level crossing methods are demonstrated in Section 1.5.

5.2 Summary of Chapter 2

In this chapter, we study the characteristics (i.e., the fluid level, the wet period, number of peaks in a wet cycle, etc.) of a fluid queue driven by an $M/G/1$ queue with *net* input rate r_1 and continuous leaking rate r_2. We remark that the fluid queue driven by an $M/G/1$ queue was first studied by Virtamo and Norros in [44] with $r_1 = r_2 = 1$ and service time of the $M/G/1$ queue being exponentially distributed. This model is re-investigated by Adan and Resing [4] using a regenerative process approach with general service time for the driving queue. The Beneš-like series of the pdf of fluid level is derived in Section 2.2.4 [7, 24]. In Section 2.3, the LST of the pdf of the wet period is derived. In Section 2.4, the characteristics of the number of tagged arrivals, arrivals served, and peaks in a wet period are discussed. In Section 2.5, simulations are conducted to verify our results on the LST of the pdf of the fluid level and the expected value of wet period. In Section 2.6, we discuss the relationship between the pdf of the fluid level and the pdf of the virtual waiting time of an $M/G/1$ queue. Later in the same section, we derive the pdf of the virtual waiting time of an $M/G/1$ queue using the pdf of the fluid level driven by an $M/G/1$ queue. Similarly, we derive the pdf of the fluid queue with balking when the fluid queue is driven by an $M/M/1/1$ queue, and use it to determine the pdf of the waiting time of the $M/G/1$ queue with balking. More importantly, Section 2.2.2 provides an alternative way, i.e. the triangle diagram, to interpret the upcrossing and downcrossing rates of level x in the fluid queue.

5.3 Summary of Chapter 3

In this chapter, we study the pdf of the fluid level in fluid models where the *net* input rate and continuous leaking rate depend on the background process, the fluid level, and the type of arrivals in the driving queue. The first fluid model in this chapter is illustrated in Section 3.1 and the second fluid model is illustrated in Figure 3.2. A triangle diagram has been applied in Section 3.2.2 to find the pdf of the fluid level in the second fluid model.

5.4 Summary of Chapter 4

In this chapter, via the level crossing methods, we derive the pdf of the fluid level that involves jump fluid inputs in the fluid queue. A pdf of the virtual waiting time for an $M/G/1$ queue with special service for zero-wait arrivals is derived using the results of the fluid queue model introduced in Section 4.1.

Appendices

A Alternative proofs of some formulas

A.1 Proof of the LST in formula (2.40)

Here, we provide alternative way to derive the expression (2.40) on page 35.

Laplace-Stieltjes transform of pdf of fluid level using the Benes-like series:

Denote by $\mathcal{L}(\bullet)$ the LST operator and by $\widetilde{C}(s)$ the LST of the fluid level. Taking the LST of the left-hand side of equation (2.31), one gets

$$\mathcal{L}_f(s) = \widetilde{\mathcal{C}}(s) - \pi_{\mathcal{E}}, \quad x > 0. \tag{A.1}$$

Similarly, taking the Laplace transform of the right-hand side of equation (2.31), one gets

$$\mathcal{L}_f(s) = \pi_{\mathcal{E}} \left(\frac{r_1 + r_2}{r_1^2} \right) r_1 \sum_{k=1}^{\infty} \left(\frac{r_1}{r_2} \rho_B \frac{(1 - \widetilde{B}(r_1 s))}{s r_1 \mathbb{E}[B]} \right)^k, \quad x > 0. \tag{A.2}$$

From (A.2), one gets

$$\mathcal{L}_f(s) = \pi_{\mathcal{E}} \frac{\lambda (r_1 + r_2)(1 - \widetilde{B}(r_1 s))}{r_1 r_2 s - r_1 \lambda (1 - \widetilde{B}(r_1 s))} \tag{A.3}$$

by recognizing that the infinite sum can be further simplified by using a geometric series argument because we know that from the restriction of having a stable fluid queue, we have

$$r_1\mathbb{E}[I] < r_2\mathbb{E}[B] \implies \frac{r_1}{r_2}\rho_B < 1. \tag{A.4}$$

Also the term

$$\frac{(1-\widetilde{B}(r_1 s))}{s r_1 \mathbb{E}[B]} \tag{A.5}$$

in equation (A.2) is the LST of a residual busy period, which is non-negative and takes the value at most 1. Thus the product of equation (A.4) and (A.5) is always less than 1 but greater than 0. Finally, we get (for $s > 0$)

$$\sum_{k=1}^{\infty}\left(\frac{r_1}{r_2}\rho_B\frac{(1-\widetilde{B}(r_1 s))}{s r_1\mathbb{E}[B]}\right)^k = \sum_{k=1}^{\infty}\left(\frac{\lambda(1-\widetilde{B}(r_1 s))}{r_2 s}\right)^k = \frac{\lambda(1-\widetilde{B}(r_1 s))}{r_2 s - \lambda(1-\widetilde{B}(r_1 s))}. \tag{A.6}$$

Setting equation (A.1) equal to equation (A.3) and solving for $\widetilde{\mathcal{C}}(s)$ leads to

$$\widetilde{\mathcal{C}}(s) = \pi_{\mathcal{E}}\left(\frac{r_1 r_2 s + r_2\lambda(1-\widetilde{B}(r_1 s))}{r_1 r_2 s - r_1\lambda(1-\widetilde{B}(r_1 s))}\right), \quad s > 0, \tag{A.7}$$

which is the LST achieved in equation (2.40).

A.2 Proof of Theorem 2.3.2

Here, we provide an alternative way to prove the *Theorem 2.3.2* stated on page 38. This theorem was first presented by Boxma et al. in [11] for a different setup of a fluid queue model that we did not discuss in this dissertation. The proof of the theorem is skipped by the authors. Thus we provide an alternative proof to verify that the theorem can be applied in our fluid queue model in Chapter 2.1.

Theorem 2.3.2: The Laplace-Stieltjes transform of the fluid wet period driven by an $M/G/1$ queue with arrival rate λ and expected service time $1/\mu$ satisfies the following functional equation

$$\widetilde{\mathcal{W}}(s) = \mathbb{E}\left[e^{-sB(1+r_1/r_2)-\lambda\left(1-\widetilde{B}(s)\right)(r_1B)/r_2}\right],$$

$$= \widetilde{B}\left(\omega\left(1+\frac{r_1}{r_2}\right)+\lambda\left(1-\widetilde{\mathcal{W}}(s)\right)\frac{r_1}{r_2}\right), \quad s>0,$$

where r_1 and r_2 are the *net* input rate and leaking rate modulated by the driving process respectively, $\widetilde{\mathcal{W}}(s)$ and $\widetilde{B}(s)$ are the Laplace-Stieltjes transforms of the fluid queue wet period and the busy period of the $M/G/1$ queue, respectively.

Proof: From equation (2.47), the wet period can be decomposed into a series of sub-wet periods. We have

$$\mathcal{W} \overset{dist.}{=} \left(1+\frac{r_1}{r_2}\right)B + \sum_{i=1}^{\mathcal{N}_T}\mathcal{W}_i,$$

where $\mathcal{N}_T$ is the number of tagged arrivals that initiate the sub-wet period of the fluid queue and $\mathcal{W}_i, i = 1, 2, \ldots,$ is the i-th sub-wet period of $\mathcal{W}$. Let $\widetilde{\mathcal{W}}(s)$, $\widetilde{B}(s)$ be the Laplace-Stieltjes transforms of the wet period of a fluid queue and busy period of an $M/G/1$ queue respectively. Taking conditional expectation of $\exp(-sB)$ on $B = x$ and

$\mathcal{N}_T = n$, and using the fact that the $\mathcal{W}_i$'s are independent and identically distributed as $\mathcal{W}$, we have

$$\begin{aligned}
\mathbb{E}[\exp(-s\mathcal{W}) \mid B = x, \mathcal{N}_T = n] &= \mathbb{E}\left[\exp\left(-s\left(1+\frac{r_1}{r_2}\right)x - s\sum_{i=1}^{n}\mathcal{W}_i\right)\right], \\
&= \left[\exp\left(-\omega\left(1+\frac{r_1}{r_2}\right)x\right)\right]\left(\widetilde{\mathcal{W}}(s)\right)^n.
\end{aligned}$$

Applying the tower property of expectation [5] on $\mathcal{N}_T = n$ yields

$$\begin{aligned}
\mathbb{E}[\exp(-s\mathcal{W}) \mid B = x] &= \sum_{n=0}^{\infty}\mathbb{E}\left[\exp\left(-s\left(1+\frac{r_1}{r_2}\right)x\right)\right]\left(\widetilde{\mathcal{W}}(s)\right)^n \mathbb{P}[\mathcal{N}_T = n] \\
&= \sum_{n=0}^{\infty}\left[\exp\left(-s\left(1+\frac{r_1}{r_2}\right)x\right)\right]\left(\widetilde{\mathcal{W}}(s)\right)^n \frac{\exp(-\frac{r_1}{r_2}\lambda x)(\lambda\frac{r_1}{r_2}x)^n}{n!} \\
&= \exp\left(-s\left(1+\frac{r_1}{r_2}\right)x - \lambda\frac{r_1}{r_2}x\right)\sum_{n=0}^{\infty}\left[\frac{\left(\widetilde{\mathcal{W}}(s)\frac{r_1}{r_2}\lambda x\right)^n}{n!}\right] \\
&= \exp\left(-\left(s\left(1+\frac{r_1}{r_2}\right)+\frac{r_1}{r_2}\lambda - \widetilde{\mathcal{W}}(s)\frac{r_1}{r_2}\lambda\right)x\right).
\end{aligned}$$

Applying the tower property again, one finally gets

$$\widetilde{\mathcal{W}}(s) = \widetilde{B}\left(s\left(1+\frac{r_1}{r_2}\right)+\lambda\left(1-\widetilde{\mathcal{W}}(s)\right)\frac{r_1}{r_2}\right), \quad \omega \geq 0.$$

$\square$

A.3 Alternative proof of integral equation in (3.13) by the method of pages

In this section, we provide an alternative proof of integral equation of the pdf of the fluid level of the fluid queue introduced in Section 3.1 by using the 'page' method (or the 'sheet' method) [12, Chapter 4, pp. 162 - 254, and Section 10.10, pp. 433 - 439].

To apply the 'page' method, we first decompose the stochastic process $\{\mathcal{C}(t), Z(t)\}_{t\geq 0}$ into 2 different sub-processes and recode the sample path of each process on 2 different 'pages', where $Z(t)$ is defined in (2.2). The sample path on 'page' 1 corresponds to the process $\{\mathcal{C}(t), Z(t) = 1\}_{t\geq 0}$, and the sample path on 'page' 2 corresponds to $\{\mathcal{C}(t), Z(t) = 2\}_{t\geq 0}$. It is important to highlight that there has no 'page' on *line* 0, since probability of the fluid level being at level 0 is 0 (see Section 3.1.2 for the details). We refer to the page contains the original sample path of $\{\mathcal{C}(t)\}_{t\geq 0}$ as a cover page. Figure A.1 illustrates a typical sample path of the $\{\mathcal{C}(t), Z(t)\}_{t\geq 0}$ and the sub-processes of it.

Page equations:

Denote by k_1 the *net* input rate of the fluid queue and by $k_2 x$ the *continuous* leaking rate of the fluid queue. The LC equations can be achieved by using 'page' method.

Page 1: The sample path rate *into* $((x,\infty),1)$ is

$$k_1 f_1(x) + \lambda \int_{y=x}^{\infty} f_2(y)dy. \tag{A.8}$$

The first term $k_1 f_1(x)$ in (A.8) is the rate at which the leading point of the sample path enters $((x,\infty),1)$ at level x from below. The second term $\lambda\int_{y=x}^{\infty} f_2(y)dy$ is the rate at which an activity period is initiated by a zero-wait arrival while the fluid level is above level x (by making instantaneous parallel transitions from $((x,\infty),2)$ into $((x,\infty),1)$).

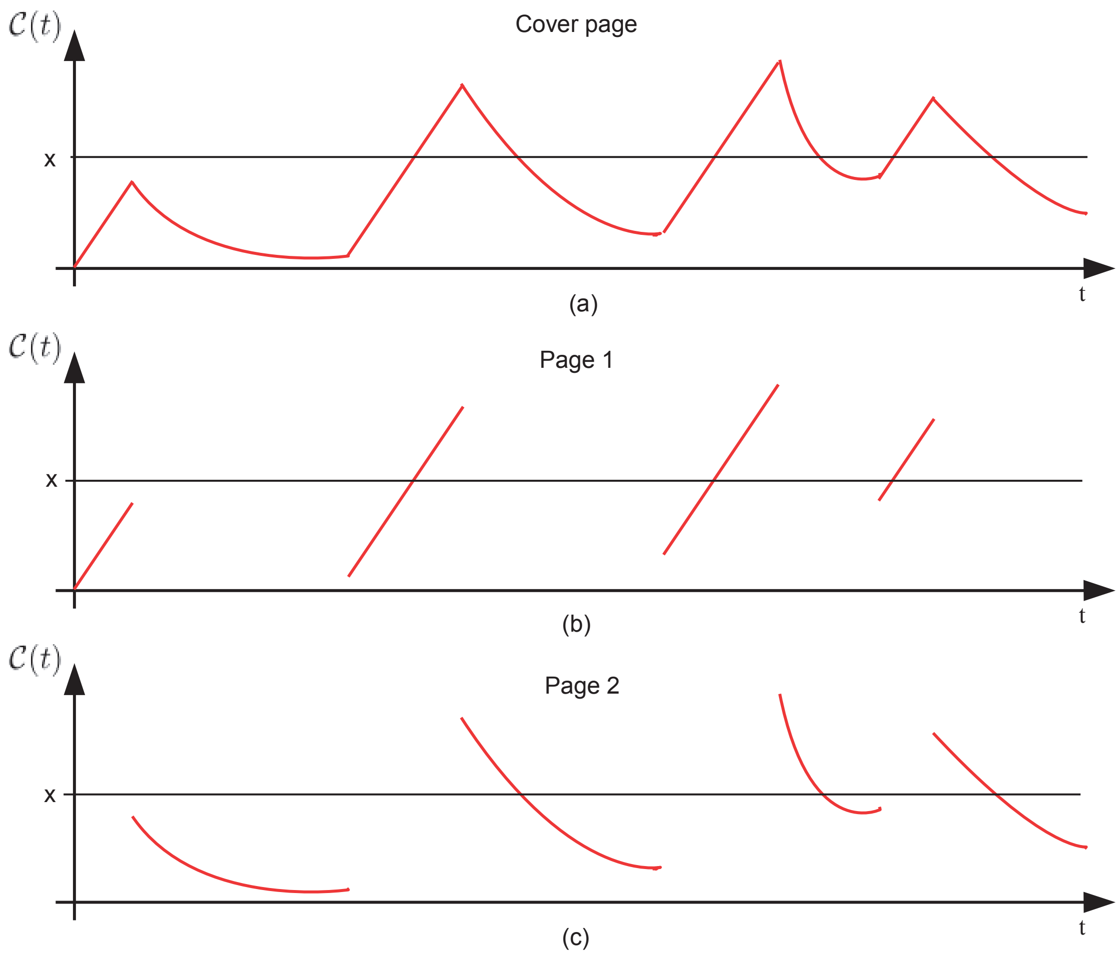

Figure A.1: A typical sample path of (a) the fluid level of the fluid queue with *continuous* leaking rate depending on fluid level, (b) the fluid level of the fluid queue on page 1, and (c) the fluid level of the fluid queue on page 2.

The sample path rate *out of* $((x,\infty),1)$ is

$$\lambda \int_{y=x}^{\infty} f_2(y)dy + \lambda \int_{y=0}^{x} \overline{B}\left(\frac{x-y}{k_1}\right) f_2(y)dy. \tag{A.9}$$

The first term $\lambda \int_{y=x}^{\infty} f_2(y)dy$ in (A.9) is the rate at which an activity period is initiated at some level above x (such that the activity period ends above level x as well). The second term $\lambda \int_{y=0}^{x} \overline{B}\left(\frac{x-y}{k_1}\right) f_2(y)dy$ in (A.9) is the rate at which an activity period is initiated at as some level below x and the duration of the busy period of the driving queue is long enough to make the sample path of the fluid level upcrosses level x. These two terms are due to instantaneous parallel transitions from 'page 2' to 'page 1'. More importantly, as highlighted in Section 3.1.2 that the $\pi_{\mathcal{E}} = \mathbb{P}[\mathcal{C} \leq 0] = 0$, there has no *upcross of level* x starting at level 0. Thus, the expressions (A.8) and (A.9) do not include any term associated with $\pi_{\mathcal{E}}$.

Balancing the sample path rate *into* and rate *out of* $((x,\infty),1)$ yields

$$k_1 f_1(x) = \lambda \int_{y=0}^{x} \overline{B}\left(\frac{x-y}{k_1}\right) f_2(y)dy. \tag{A.10}$$

Page 2: Similarly, the sample path rate *into* $((x,\infty),2)$ is

$$\lambda \int_{y=x}^{\infty} f_2(y)dy + \lambda \int_{y=0}^{x} \overline{B}\left(\frac{x-y}{k_1}\right) f_2(y)dy, \tag{A.11}$$

where the sample path rate *out of* $((x,\infty),2)$ is

$$k_2 x f_2(x) + \lambda \int_{y=x}^{\infty} f_2(y)dy. \tag{A.12}$$

The interpretation of (A.11) and (A.12) is similar to (A.9) and (A.8), respectively. Balancing the sample path rate *into* and rate *out of* $((x,\infty),2)$ yields

$$\lambda \int_{y=0}^{x} \overline{B}\left(\frac{x-y}{k_1}\right) f_2(y)dy = k_2 x f_2(x). \tag{A.13}$$

Summing the expressions (4.41) and (4.44) yields

$$k_1 f_1(x) = k_2 x f_2(x) \implies f_1(x) = (k_2 x f_2(x))/k_1 \implies f_2(x) = (k_1 f_1(x))/(k_2 x). \tag{A.14}$$

The expression (A.14) is achieved in (3.3) using different approach.

Substituting $f_2(x) = (k_1 f_1(x))/(k_2 x)$ into (A.10) and dividing both sides of (A.10) by k_1

gives

$$f_1(x) = \lambda \int_{y=0}^{x} \frac{1}{k_2 y} \overline{B} \left(\frac{x-y}{k_1} \right) f_1(y) dy. \tag{A.15}$$

Dividing both side of (A.12) by $k_2 x$ yields

$$f_2(x) = \frac{\lambda}{k_2 x} \int_{y=0}^{x} \overline{B} \left(\frac{x-y}{k_1} \right) f_2(y) dy. \tag{A.16}$$

Summing the expressions (A.15) and (A.16) gives the integral equation of the pdf of the fluid level, namely

$$f(x) = \frac{\lambda(k_1 + k_2 x)}{k_2 x} \int_{y=0}^{x} \overline{B} \left(\frac{x-y}{k_1} \right) \frac{1}{k_1 + k_2 y} f(y) dy. \tag{A.17}$$

The expression (A.17) is achieved in (3.13) using different approach. Using two different approaches, we achieved the same integral equation expression for the pdf of the fluid level, it increases the confidence that the integral equation expression of the pdf of the fluid level $f(x)$ is correct.

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